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Essential Singularity in the Percolation Model

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It is rigorously proved that the analog of free energy for the percolation models has a singularity at zero external field as soon as percolation appears. The singularity is an essential one at least for large concentrations. Results on the asymptotic behavior of the cluster-size distribution are also obtained for percolation models and for the Ising model at low temperatures.

In recent years, the ideas and results of percolation theory have found wide use in the physics of disordered systems.[1] In particular, the analogy with second-order phase transitions has been elucidated, following the work of Fortuin and Kasteleyn.[2] What resulted from these investigations is that a quantity analogous to the free energy of one phase can be defined for the percolation model on any lattice by

$$f_p(h)=\sum_{n=1}^{\infty}\frac{1}{n}P_n(p)e^{-hn},$$

where p is the probability that each site of the lattice be occupied, independently of the other sites, h a parameter playing the role of the external magnetic field in ordinary phase-transition problems, and $P_n(p)$ is the probability that the origin belongs to a cluster of size n. A cluster is a set of occupied sites connected by the bonds of the lattice, and completely surrounded by empty sites. Moreover, $f_p(h)$ is evidently the generating function of the various moments of the cluster-size distribution which are of physical interest in the percolation problem.

Since $f_p(0)\leq 1$, the function $f_p(h)$ is analytic in h for $\mathrm{Re}h\geq 0$, and an interesting question is therefore to understand the connection between the appearance of an infinite cluster above the percolation threshold p_c, the analytic properties of $f_p(h)$ at $h=0$, and the behavior of the cluster-size distribution P_n and its moments $\langle |C|^n\rangle$. What we have done is to prove rigorously that these quantities have qualitatively different behaviors below and over p_c.

Our results for the percolation problem on the square lattice in ν dimension ($\nu>1$) are summarized in the following theorem: (1) For $p<p_0(\nu)<p_c(\nu)$, $f_p(h)$ is analytic at $h=0$, P_n decays exponentially, and $\langle |C|^n\rangle$ behaves as $K_1^n n!$. (2) For $p>p_c(\nu)$, $f_p(h)$ is singular at $h=0$, P_n does not decay exponentially, and $\langle |C|^n\rangle\geq K_2^n[\nu/(\nu-1)n]!$. (3) For $p>p_1(\nu)>p_c(\nu)$, the singularity of $f_p(h)$ at $h=0$ is an essential one and

$$\exp(-\alpha n^{(\nu-1)/\nu})\leq P_n\leq\exp(-\alpha' n^{(\nu-1)/\nu}),$$

$$K_3^n\left(\frac{\nu}{\nu-1}n\right)!\leq\langle |C|^n\rangle\leq K_4^n\left(\frac{\nu}{\nu-1}n\right)!.$$

We will first give the intuitive basis of the proof, which is by itself of interest for an understanding of the physical phenomenon involved, then we will sketch the proof, and finally we will discuss these results.

The intuitive basis of the proof is the following: In very large but finite clusters, point which are far from the external boundary can be connected to it above the percolation threshold. Therefore, the effective volume of the clusters with a given external boundary should be of the order of the geometric volume included inside the boundary, multiplied by the percolation probability. However, below the percolation threshold, the effective volume should be only of the order of the boundary. Let us now sketch the proof of part (2) of our theorem. Part (1) is obtained through a bound on the number of clusters of size n, and part (3) by an elaboration of the methods used for part (2). Detailed proofs and other results will be given elsewhere.[3]

For simplicity we restrict ourselves to the two-dimensional square lattice Z^2. If $q=1-p$, the "free-energy density" $f_p(h)$ can be written as

$$f_p(h)=\sum_{C\ni\{0\}}\frac{e^{-h|C|}}{|C|}p^{|C|}q^{|\partial C|}=\sum_{C_1}e^{-h|C|}p^{|C|}q^{|\partial C|}, \qquad (1)$$

where the sum runs over all the finite clusters C containing the origin in the first expression and over all possible shapes of finite clusters in the second one. $|C|$ denotes the size of the cluster

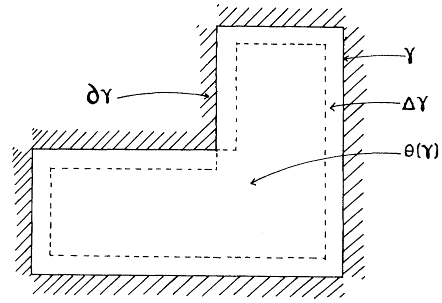

FIG. 1. A contour γ.

C and $|\partial C|$ the size of its boundary, i.e., the number of points of $Z^2\backslash C$ which are nearest neighbors to some point of C.

In view of the positivity of the coefficients in the series (1), the singularity at $h=0$ will imply the nonexponential decay of P_n and can be obtained from a lower bound on the moments so that

$$\langle |C|^{n-1}\rangle=\sum_{C_1}|C|^n p^{|C|}q^{|\partial C|}. \qquad (2)$$

The contribution of each cluster will be separated into a surface effect and a volume effect. We associate therefore to each cluster C a contour γ, corresponding to the outer order border of C; $V(\gamma)$ are the points of Z^2 inside γ, $\Delta\gamma$ the set of points in $V(\gamma)$ nearest neighbors to some point of the outside of γ, and $\theta(\gamma)$ is $V(\gamma)\backslash\Delta\gamma$ (see Fig. 1). Hence, we can rewrite (2) as

$$\langle |C|^{n-1}\rangle=\sum_{\gamma_1}p^{|\Delta\gamma|}q^{|\partial\gamma|}\sum_{\substack{E\subset\theta(\gamma)\\ E^C/\Delta\gamma}}(|\Delta\gamma|+|E|)^n p^{|E|}q^{|\partial E\cap\theta(\gamma)|}, \qquad (3)$$

where the sum on E runs over all subsets of $\theta(\gamma)$ such that the points of E and of $\Delta\gamma$ are connected.

Now Eq. (3) can be transformed into

$$\langle |C|^{n-1}\rangle=\sum_{\gamma_1}p^{|\Delta\gamma|}q^{|\partial\gamma|}\sum_{m=0}^{n}\binom{n}{m}|\Delta\gamma|^{n-m}\sum_{\substack{x_1\in\theta(\gamma),\\ \dots\\ x_m\in\theta(\gamma)}}\alpha_{\theta(\gamma)}(x_1,\dots,x_m), \qquad (4)$$

where $\alpha_{\theta(\gamma)}(x_1,\dots,x_m)$ denotes the probability inside the box $\theta(\gamma)$ that the points $x_1,\dots,x_m$, possibly identical, of $\theta(\gamma)$ and the points of $\Delta\gamma$ are all connected. In the derivation of Eq. (4) from Eq. (3), we have used the probabilistic interpretation of $p^{|E|}q^{|\partial E\cap\theta(\gamma)|}$.

We will restrict now the summation in (4) to the contours γ_l which are the squares of side l; $\Delta\gamma$ is then by itself a connected set and if $\chi_{x_i}{}^{\Delta\gamma}$ denotes the characteristic function of the event: "x_i is connected to the boundary $\Delta\gamma$ of $\theta(\gamma)$" so that

$$\alpha_{\theta(\gamma)}(x_1,\dots,x_m)\left\langle\prod_{i=1}^{m}\chi_{x_i}{}^{\Delta\gamma}\right\rangle_{\theta(\gamma)}.$$

Now from Harris[2] or Fortuin, Ginibre, and Kasteleyn[4] we know that if $g_i,\dots,g_m$ are m increasing functions over the configurations, that is $g_i(X)\geq g_i(X')$ whenever the set of occupied points of a configuration X contains the set of occupied points of the configuration X', then

$$\langle g_1,\dots,g_m\rangle\geq\prod_{i=1}^{m}\langle g_i\rangle.$$

Since the functions $\chi_{x_i}{}^{\Delta\gamma}$ are increasing functions over the configurations of $\theta(\gamma)$, we get

$$\alpha_{\theta(\gamma)}(x_1,\dots,x_m)\geq\prod_{i=1}^{m}\alpha_{\theta(\gamma)}(x_i).$$

But $\alpha_{\theta(\gamma)}(x_i) \geq P_\infty$, where P_∞ denotes the percolation probability and so we obtain

$$\langle |C|^{n-1}\rangle \geq \sum_{\gamma_l} p^{|\Delta\gamma|} q^{|\partial\gamma|}(|\Delta\gamma| + P_\infty|\theta(\gamma)|)^n. \quad (5)$$

The term $l=n$, for example, leads to $\langle |C|^{n-1}\rangle \geq K_1^n(2n)!$ whenever $P_\infty > 0$. The terms which create the divergence are those coming from $P_\infty|\theta(\gamma)|$. They represent, as claimed previously, the contribution of the effective volume of the clusters.

Concerning the percolation problem, our results give a detailed description of the behavior of the cluster distribution function. In particular the result on the moments implies that if $P_n \sim \exp(-\alpha n^\xi)$ then for $p > p_c$, $\xi \leq (\nu-1)/\nu$. Moreover, exact behavior of P_n is obtained rigorously for large concentration. Our theorem generalizes to other "realistic" lattices such as the triangular, hexagonal, fcc lattices. In contrast, for the Bethe tree, P_n decays exponentially everywhere, except at p_c.[5] In fact, our results contradict some of the claims made in the literature, but they also yield a rigorous basis to some conjectures by Stauffer[6] obtained from an analysis of numerical studies.

Our result can be generalized to the case of the interacting percolation problem where the probability of occupation of each site is no longer independent but is given through the probability distribution of some equilibrium system of statistical mechanics. As an example, we obtain that for a wide class of equilibrium states, including the Ising model, P_n no longer decays exponentially as soon as there is percolation.

The situation that we get for the cluster generating function has to be paralleled with the Andreev-Fisher conjecture[7] on the existence of an essential singularity of the free energy at phase transition points. Our result can be seen as a rigorous proof of the analog conjecture for the percolation problem. However, one has to remark that in the case of Andreev and Fisher, for example, in the low-temperature Ising model, one has a reasonable approximation of the free energy by restricting to the compact clusters, which are the first term in a low-temperature expansion, and this yields a singularity. There the problem is then to know whether going beyond the approximation would make the singularity disappear. In contrast, in the percolation problem, one does not have such a reasonable approximation, and in fact the contribution of the compact clusters to the cluster generating function yields an approximation to $f_p(h)$ which is analytic and not singular at $h=0$ as can be seen by explicit computation. In percolation, the singularity does not come from a special simple class of clusters but only from an average over the clusters with a same external boundary. This average yields an effective volume to the clusters over the percolation threshold as we have proved.

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