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Boojums in Superfluid ^{3}He-*A* and Cholesteric Liquid Crystals

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Because of the similarity of their order parameters, there are close analogies between defects of ^{3}He-*A* and cholesteric liquid crystals. In particular, boojums, originally predicted for ^{3}He-*A*, should exist as well in cholesterics. Certain textures experimentally observed and reported in the literature are identified as boojums. A topological analysis is given, and the effects of boojums on dynamical properties of cholesterics are discussed.

It is known that the topological characteristics of the boundary of a system with spontaneously broken symmetry are connected to the appearance of singularities in the order parameter. It follows that in order to fully treat the problem of textures in the bulk, one must consider global boundary conditions as well as the topology of the order parameter. This approach has been of great interest recently in determing the properties of ^{3}He-*A* in containers of various topologies.[1-3] One of the most interesting conclusions drawn from these studies is the necessity of new types of surface singularities in the order parameter of ^{3}He-*A* in certain containers. For example, if ^{3}He-*A* is placed in a sphere subject to the boundary condition that the two gap-parameter vectors $\vec{\Delta}_1$ and $i\vec{\Delta}_2$ are constrained to lie in the surface, a singularity in the pair angular momentum vector $\vec{l}$ must appear, and it is believed that the lowest-energy configuration consists of an isolated singularity lying on the surface of the sphere. This type of singularity is known as a boojum[3] (see Fig. 1), and connects to a nonsingular vortex texture in the bulk. The appearance of a boojum in a sample of ^{3}He-*A* can lead to decay of superflow without the nucleation of highly singular vortex-line cores.[3,4] It there-

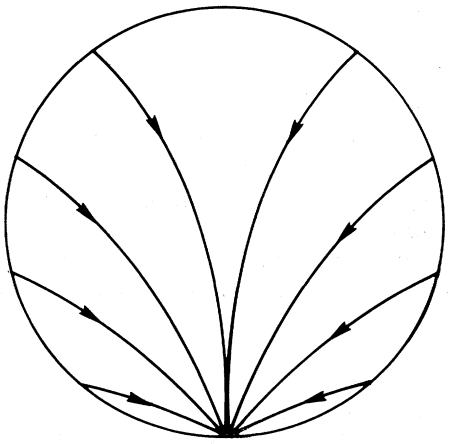

FIG. 1. A simple spherical boojum. Lines represent $\vec{l}$ in ^{3}He-*A*, $\vec{t}$ in cholesterics.

fore appears that a thorough study of these structures is of fundamental importance in understanding both static and dynamical properties of ^{3}He-A. In this Letter, we point out that these structures are to be expected in a different system, that is, cholesteric liquid crystals, that in fact they have already been experimentally observed, and that cholesterics provide a very convenient "tabletop" laboratory for examining various properties of boojums.

The similarity between ^{3}He-A and cholesteric liquid crystals can be traced to the fact that their order parameters look very much alike. It has been proposed that the order parameter of a cholesteric consists of an orthonormal triad of vectors[5-8]: the twist axis $\vec{t}$, the directrix $\vec{d}$, and $\vec{t}\times\vec{d}$, where none of these vectors has a meaningful sense. We wish, however, to point out that there may be problems in applying topological analysis with a nonlocal order parameter.[9] The order parameter in the bulk liquid is the coset space $SU(2)/Q$, where Q is the group of quaternions. This gives rise to the appearance of eight classes of line defects, including the defect-free state[6,7]; however, because of the non-Abelian nature of Q, line defects are classified according to the five conjugacy classes of Q: 1, -1, $\pm i\sigma_x$, $\pm i\sigma_y$, and $i\sigma_z$, where σ_i is a Pauli matrix. Point and planar defects are topologically unstable in the bulk; they can be continuously deformed into no defect.

If a cholesteric with this order parameter is placed in a container subject to the constraint that $\vec{d}$ is everywhere parallel to the surface of the container, singularities must appear in $\vec{d}$ if the Euler characteristic E of the surface is nonzero. Mermin,[2] in the corresponding case of ^{3}He-A, first proposed the equation

$$E=\sum_{\substack{\text{all}\\ \text{vortices}}} m\,, \tag{1}$$

where m is the strength of a vortex terminating in the surface; a general discussion will be given, following the works of Anderson and Palmer[1] and of Mermin.[3]

For a smooth surface of genus n (that is, a pretzel with n holes), Euler's characteristic is given by

$$E=2(1-n) \tag{2}$$

and so a sphere has $E=2$, a torus $E=0$, etc. Note that this implies that a torus is the only geometry capable of sustaining no singularities. A sphere, on the other hand, must sustain a total vorticity of 4π. A way to accomplish this consists of four $m=\frac{1}{2}$ or two $m=1$ vortices terminating on the surface; the bending energy per unit length of a disgyration of strength m in a cholesteric is given by[5]

$$g_L \sim m^2 \ln(R/r_0)\,, \tag{3}$$

where R is the size of the container and r_0 the width of the line defect. Therefore, one solution of the problem of a cholesteric in a sphere is line vortices terminating on the surface. However, another solution exists, which entails the collapse of the vortices into a single, isolated point singularity on the surface—that is, the boojum of Fig. 1! This is discussed in detail for ^{3}He-A by Anderson and Palmer[1] and by Mermin.[3] It is easily seen that the boojum configuration is energetically preferred to line vortices, the bending energy of the former being much less. If the solid lines in Fig. 1 represent the twist axes $\vec{t}$, the directrices $\vec{d}$ will appear as in Fig. 2. This is the solution of the problem of a cholesteric in a sphere. It should be noted that under certain deformations of the sphere, vortices may have the lower energy.

These considerations are now made rigorous by examining the topological stability of surface defects. Using the Toulouse-Kleman theory,[10] we surround a point defect on a two-dimensional manifold by a circle, a line by two points, etc., and then examine the homotopy class of the map from physical space to order-parameter space.

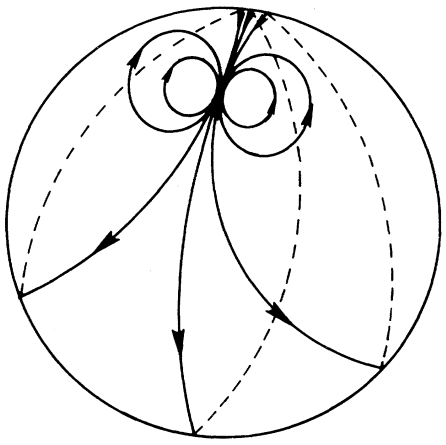

FIG. 2. Directrix lines $\vec{d}$ on a spherical surface for the boojum in Fig. 1. The point of tangency of all circles on the sphere corresponds to the singularity at the south pole in Fig. 1. As one moves into the interior, the circles slightly separate from the point of tangency and rotate about the radius.

For cholesterics the order parameter on a surface is the quotient group $SO(2)/Z_2$. Its first two homotopy groups are

$$\pi_0[SO(2)/Z_2] = 0, \quad (4)$$

$$\pi_1[SO(2)/Z_2] = Z, \quad (5)$$

so that there are no topologically stable line defects and an infinite, though countable, number of topologically distinct surface point defects. These point defects may be characterized by a half-integer m, and add isomorphically to the integers. In this case, Eq. (1) holds if we interpret m as the sum of the orders of surface point defects.

In contrast, the order parameter of ^{3}He-A on a surface is $O(2) = SO(2) \times Z_2$, so that

$$\pi_0[O(2)] = Z_2, \quad (6)$$

$$\pi_1[O(2)] = Z. \quad (7)$$

From Eq. (7) it is clear that point defects on a surface are topologically analogous for ^{3}He-A and cholesterics. The interesting new information is contained in Eq. (6), which states that stable line singularities on the surface may indeed exist in this case; topologically, only one kind of line defect may be present, and it is its own "antiparticle." This line defect is the border of an "island of reversed $\vec{l}$"[1,3] which constitutes a third solution to the problem of ^{3}He-A in a sphere, the other two being analogous to those discussed previously for cholesterics. Here Eq. (1) cannot be taken literally, since any "island" can substitute for a vortex point with $m = \pm 2$.

As yet no positive identifications of a boojum in ^{3}He-A have been made. It is thus of great interest to note that boojums have in fact been seen in cholesterics, although they have not previously been identified as such. In a paper appearing some twenty years ago, Robinson, Ward, and Beevers[11] observed that when the cholesteric poly-Γ-benzyl-L-glutamate (PBLG) was dissolved in any of several media (e.g., dioxan), two phases—one birefringent and one isotropic—could exist in equilibrium. The interfacial tension between the two phases caused spherical droplets ("spherulites") of the birefringent phase to become dispersed throughout the isotropic phase. This is then just the problem of a cholesteric in a sphere, and we would expect that the directrices would line up on the surface as in Fig. 2. This, in fact, is exactly what was observed.

This surface texture is identical to that of the Frank-Pryce texture,[11] which is analogous to that of the vorton[12] in ^{3}He-A; in the Frank-Pryce texture, an $m = 2$ vortex line extends from the center of the sphere to the surface, with the directrix texture of Fig. 2 twisting about this line as we move from the surface to the center. The vorton in ^{3}He-A is believed to be unstable[3]; it collapses to the boojum. Similarly, in cholesterics we do not expect such a texture to be realized, since point singularities in the bulk are topologically unstable, as previously noted, and the vorton has a higher bending energy than the boojum. We therefore expect that at least the larger spherulites are boojums.

It appears, then, that new possibilities abound for the experimental study of boojums. Since it is much easier (and less expensive) to prepare a sample of cholesteric within a smooth container of Euler invariant E than it is to prepare a specimen of ^{3}He-A, cholesterics provide us with a unique and accessible means of probing many of the properties of boojums. These results can be of great importance in understanding many of the properties of ^{3}He-A. However, boojums in cholesterics are interesting in their own right. For example, they will contribute to dissipation in order-parameter motion in cholesterics, as a classical analog to their role in ^{3}He-A. More importantly, they may play a fundamental role in nucleation of the cholesteric state in a nematic $\rightarrow$ cholesteric transition. This process seems likely to occur for a nematic created from a cholesteric by a magnetic field. If the field is then lowered below H_{crit}, we have what might be called a "supercooled" nematic, which at some point will spontaneously transform into a cholesteric. Friedel and de Gennes[13] proposed that this process was nucleated by a "disinclination loop" (similar to a vortex line closed in on itself) on the surface with a radius increasing in time. This will only occur if a loop is created with a radius greater than some critical value; otherwise, the loop shrinks to a point. We note that this latter process, which is likely to be energetically preferred, creates a boojum which may nucleate the transition.

It is clear that much work, both theoretical and experimental, remains to be done. On the theoretical side, the topological possibilities of cholesterics have yet to be completely worked out, and the effects of topology on static and dynamic properties need to be known. Experimentally, the appearance and properties of boojums need to be extensively examined.

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Temperature Hysteresis of the Transition between the Close-Packed Phases of ^{4}He

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The transition hcp ^{4}He to fcc ^{4}He shows temperature hysteresis. The difference between the transition temperature in heating and in cooling is 25 mK near the triple point, and it increases with increasing pressure. The transition is believed to be of martensitic type. Latent heat measurements indicate that above 16 K only partial transformation occurs.

The transition between hcp and fcc ^{4}He, discovered by Dugdale and Simon,[1] has not been very extensively studied. A single point near the triple point was reported by Dugdale and Franck.[2] Mills and Schuch[3] identified the high-temperature phase as fcc, using x-ray diffraction; the fcc phase was also studied by inelastic neutron scattering by Eckert, Thomlinson, and Shirane.[4] In later x-ray work, Mills and Schuch[5] succeeded in observing two crystals which showed both hcp and fcc reflections. These crystals were believed to lie on the transition line. Mills and Schuch suggested, based on their own work and that of Dugdale and Simon, a transition line in the P-T plane up to a pressure of 4.5 kbar, which becomes increasingly steep with rising pressure. Theoretical studies were reported by Hoover,[6] Alder, Carter, and Young,[7] and Holian *et al.*[8] These calculations succeeded in obtaining values of the transition entropy and the molar volume change in good agreement with experimental data. Holian *et al.* predict that the transition line in the P-T plane gradually steepens and above a certain temperature assumes negative slope, giving a maximum pressure at 0 K for the transition. The maximum pressure is predicted in the range 15 to 80 kbar.

In this study the transition was observed thermally from the vicinity of the triple point to about 3.7 kbar. It was found that the transition shows temperature hysteresis of about 25 mk near the triple point; with increasing pressure the hysteresis increases. The maximum so far observed is 450 mk. The general character of the observed transitions strongly suggests that the transition is of martensitic type. The slope of the transition line in the P-T plane decreases continually with rising pressure, becoming essentially linear

