predicts[15] an angular correlation with a minimum at 90° at all the measured electron scattering angles, in strong disagreement with the measurements. Again for clarity the angular correlations have not been included in Fig. 2.

Comparison of the 100-eV data with the various approximation therefore shows that although these approximations may describe the differential $2s+2p$ cross sections quite accurately, they do not give an adequate description of the coincidence measurements. The only approximation to improve noticeably on the BA description of the data is the DWPO model. The CPB, CPTM, and UGA models give much poorer results. The 70-eV data are in rather better agreement with the simple BA.

[1]M. Eminyan, K. B. MacAdam, J. Slevin, and H. Kleinpoppen, J. Phys. B 7, 1519 (1974).

[2]A. Ugbabe, P. J. O. Teubner, E. Weigold, and H. Arriola, J. Phys. B 10, 71 (1977).

[3]S. T. Hood, A. J. Dixon, and E. Weigold, to be published.

[4]E. Weigold, S. T. Hood, I. Fuss, and A. J. Dixon, J. Phys. B 10, L623 (1977).

[5]I. E. McCarthy and E. Weigold, Phys. Rep. 27C, 275 (1976).

[6]I. Fuss, I. E. McCarthy, C. J. Noble, and E. Weigold, Phys. Rev. A 17, 604 (1978).

[7]J. Macek and D. H. Jaecks, Phys. Rev. A 4, 2288 (1971).

[8]U. Fano and J. Macek, Rev. Mod. Phys. 45, 553 (1973).

[9]J. F. Williams, in *Proceedings of the Ninth International Conference on the Physics of Electronic and Atomic Collisions, Seattle, Washington, 1975*, edited by J. S. Risley and R. Geballe (Univ. of Washington Press, Seattle, 1976), pp. 138–150.

[10]R. V. Calhoun, D. H. Madison, and W. N. Shelton, J. Phys. B 10, 3523 (1977).

[11]M. J. Roberts, J. Phys. B 11, 2219 (1977).

[12]M. R. C. McDowell, L. A. Morgan, and V. P. Myerscough, J. Phys. B 8, 1053 (1975).

[13]L. A. Morgan and M. R. C. McDowell, J. Phys. B 8, 1073 (1975).

[14]J. N. Gau and J. Macek, Phys. Rev. A 12, 1760 (1975).

[15]L. A. Morgan and A. D. Stauffer, J. Phys. B 8, 2342 (1975).

[16]J. F. Williams and B. A. Williams and B. A. Willis, J. Phys. B 8, 1641 (1975).

Lie-Operator Approach to Mode Coupling in Nonuniform Plasma

Shayne Johnston
Plasma Physics Laboratory, Columbia University, New York, New York 10027

and

Allan N. Kaufman
Department of Physics and Lawrence Berkeley Laboratory, University of California, Berkeley, California 94720
(Received 31 March 1978)

Hamiltonian perturbation theory based on recent Lie transform techniques is applied to the Hamiltonian of a single particle in nonuniform Vlasov plasma. A simple relation is derived between the field-plasma interaction energy and the transformed single-particle Hamiltonian. This relation implies as special cases a general formula for ponderomotive force in terms of the linear Vlasov susceptibility, and a symmetric Poisson-bracket formula for the general three-mode coupling coefficient.

Nonlinear interaction among waves and particles in plasma occurs in problems of parametric instability[1] and weak plasma turbulence[2]; these nonlinear processes have important applications to such areas as radio-frequency plasma heating, laser-plasma coupling, and the stabilization of linear instabilities. Several alternative theoretical approaches to these problems have evolved, including a direct perturbation expansion of the governing classical equations (e.g., the Vlasov-Maxwell system[3]), the temporary introduction of quantum mechanical ideas,[4] and the averaged-Lagrangian method.[5] Recently, the present authors have suggested a new approach[6-8] based upon a canonical transformation[9] of the single-particle Hamiltonian. This viewpoint, named the method of generalized ponderomotive forces, has been shown to provide a systematic and intuitive framework for the study of mode coupling in magnetized Vlasov plasma[8]; among its advantages are the elegance and efficiency of Hamiltonian perturbation theory, and an appealing decomposition of the physical current density. In this article, we re-examine our Hamiltonian approach in the light of recent Lie-operator formulations, and demonstrate the remarkable ease with which certain simple and general results can be derived.

The coupling of linear modes in plasma is essentially a perturbative notion, and, if one is committed to doing a perturbation theory, then it is reasonable to seek the most compact and systematic version of it available. Workers in celestial mechanics have become experts on this subject in computing the orbits of heavenly bodies. In particular, Hamiltonian perturbation theory[10] has been significantly refined in the last decade through the introduction of Lie transform techniques.[11] The Lie transform approach is usually applied to Hamiltonian systems depending on a parameter ϵ; the solution when $\epsilon = 0$ is assumed known. A canonical transformation which solves the modified ($\epsilon \neq 0$) system is then constructed as a power series in ϵ. Although little is known about the general convergence properties of such series, they have proved very useful in applications. An important feature of the Lie transform approach is an avoidance of the usual mixing[12] of old and new variables.

Recently, Dewar[13] has developed another operator formalism for canonical transformations depending on a parameter ϵ, and shown it to be equivalent to that of Deprit.[11] An important virtue of Dewar's formulation is the derivation of a *nonperturbative* form of the Hamilton-Jacobi equation for the Lie generating function W, viz.,

$$\partial_t W + \{W, K\} = G_W \partial H/\partial\epsilon - \partial K/\partial\epsilon, \tag{1}$$

where the braces denote the Poisson-bracket operation. Equation (1) describes a canonical transformation $(\vec{q}, \vec{p}, H) \to (\vec{Q}, \vec{P}, K)$ of a given Hamiltonian system $H(\vec{q}, \vec{p}, t; \epsilon)$. The old and new coordinates are related by the canonical transformation operator G_W according to $\vec{q} = G_W \vec{Q}$ and $\vec{p} = G_W \vec{P}$, where the operator G_W is defined by the conditions $\partial G_W/\partial\epsilon = L_W G_W$, $G_W(\epsilon = 0) = 1$, and hence

$$G_W = 1 + \int_0^\epsilon d\epsilon_1 L_W(\epsilon_1) + \int_0^\epsilon d\epsilon_1 \int_0^{\epsilon_1} d\epsilon_2 L_W(\epsilon_1) L_W(\epsilon_2) + \cdots,$$

where L_W denotes the Lie-derivative operator $\{\ , W\}$. Equation (1) is, in principle, a nonperturbative relation; in practice, it can be employed to arbitrary order in ϵ by power-series expansion.

Let us apply this formalism to the Hamiltonian of a single particle in hot collisionless plasma. Striving for generality, we permit the plasma to be nonuniform, bounded, and relativistic, and the fields in the plasma to be nonuniform and electromagnetic. Accordingly, for arbitrary gauge, we write the electromagnetic potentials in the form

$$\vec{A}_{\rm tot}(\vec{x}, t) = \vec{A}_0(\vec{x}) + \epsilon \vec{A}(\vec{x}, t),$$
$$\varphi_{\rm tot}(\vec{x}, t) = \varphi_0(\vec{x}) + \epsilon\varphi(\vec{x}, t),$$

considering $(\vec{A}, \varphi)$ as a perturbation of the static equilibrium potentials $(\vec{A}_0, \varphi_0)$. From the interaction Lagrangian for a single particle, we then obtain[14] the relation

$$\partial H/\partial\epsilon = e\varphi - (e\vec{A}/c)(\partial H/\partial\vec{p}), \tag{2}$$

which is valid even for a relativistic particle. Now, the charge and current densities in the plasma can be written in terms of the one-particle distribution function $f(\vec{r}, \vec{p}, t)$ as

$$\rho(\vec{x}, t) = e\int d\Gamma\, \delta(\vec{x} - \vec{r}) f(\vec{r}, \vec{p}, t),$$
$$\vec{J}(\vec{x}, t) = e\int d\Gamma\, \delta(\vec{x} - \vec{r}) f(\vec{r}, \vec{p}, t) \partial H(\vec{r}, \vec{p}, t)/\partial\vec{p},$$

where $d\Gamma = d^3r\, d^3p$. Thus, evaluating the field-plasma interaction energy and noting relation (2), we find

$$\int d^3x (\rho\varphi - c^{-1}\vec{J}\cdot\vec{A}) = \int d\Gamma\, f \partial H/\partial\epsilon. \tag{3}$$

Suppose we now perform an arbitrary canonical transformation. The Vlasov equation for new entities will be

$$\partial_t F = \{K, F\}, \tag{4}$$

where $F = G_W f$. Since G_W is a unitary operator, manipulation of the right-hand side of (3) yields

$$\int d\Gamma\, f \frac{\partial H}{\partial\epsilon} = \int d\Gamma\, (G_W^{-1} F) \frac{\partial H}{\partial\epsilon} = \int d\Gamma\, F G_W \frac{\partial H}{\partial\epsilon},$$

leaving us in a position to exploit the Hamilton-Jacobi equation (1). Indeed, upon replacing $G_W \partial H/\partial\epsilon$ by (1), and using partial integration and (4) to rewrite the Poisson-bracket term, we obtain the simple and general relation

$$\int d^3x (\rho\varphi - c^{-1}\vec{J}\cdot\vec{A}) = \int d\Gamma \left[F \frac{\partial K}{\partial\epsilon} + \partial_t(FW)\right]. \tag{5}$$

Note that Eq. (5) is nonperturbative in ϵ and that the particular canonical transformation has not yet been specified.

We shall discuss two applications of relation (5). The first concerns a general formula[15] for the ponderomotive (quasistatic) force exerted on the oscillation center of a particle in a high-frequency field. Adopting the radiation gauge $\varphi = 0$, let us devise a canonical transformation to eliminate all *linear* terms in the perturbed Hamiltonian:

$$K = H_0 + K^{(2)} + O(\epsilon^3),$$
$$K^{(1)} = 0, \quad F = f_0 + O(\epsilon^2);$$

this transformation can be effected provided that any field-particle resonances are neglected.[16] The static component of (5) then yields, correct to order ϵ^2,

$$-c^{-1}\int d^3x \langle \vec{J}^{(1)} \cdot \vec{A}\rangle = 2\int d\Gamma\, f_0 \langle K^{(2)}\rangle,$$

i.e., a relation between the linear current density $\vec{J}^{(1)}$ and the ponderomotive Hamiltonian $\langle K^{(2)}\rangle$ of an oscillation center. Introduction of the linear susceptibility tensor $\overleftrightarrow{\chi}_\omega$ and functional differentiation with respect to f lead at once to the formula

$$\langle K^{(2)}(\Gamma)\rangle = -(4\pi)^{-1}\int d^3x \int d^3x' \vec{E}_\omega{}^*(\vec{x})\vec{E}_\omega(\vec{x}') : \delta\overleftrightarrow{\chi}_\omega(\vec{x}, \vec{x}')/\delta f(\Gamma),$$

which was presented (without derivation) by Cary and Kaufman.[15]

Our second application of (5) concerns the resonant nonlinear coupling of three modes of the form $\vec{A}_a(\vec{x}) \exp(-i\omega_a t)$ with $\omega_1 + \omega_2 \approx \omega_3$. If $\vec{J}_a{}^{(2)}$ denotes the nonlinear current density at frequency ω_a due to the beating of modes b and c, and if $\mathcal{E}_a$ denotes the total energy in mode a, then the equations governing action transfer among the modes and the frequency shift of each mode are,[17] respectively,

$$\frac{1}{\omega_a}\frac{d\mathcal{E}_a}{dt} = 2\,\mathrm{Im}U_a, \quad \frac{\delta\omega_a}{\omega_a} = \mathcal{E}_a{}^{-1}\,\mathrm{Re}U_a,$$

where we have defined the coupling coefficient

$$U_a = -c^{-1}\int d^3x\, \vec{J}_a{}^{(2)}(\vec{x}) \cdot \vec{A}_a{}^*(\vec{x}).$$

Using the method of generalized ponderomotive forces, we have previously evaluated the coefficients U_a and shown explicitly that $U_1 = U_2 = U_3{}^*$, obtaining a symmetric Poisson-bracket formula for the coupling coefficient U[7,17]; this symmetry implies the Manley-Rowe relations[18]

$$\omega_1{}^{-1}\frac{d\mathcal{E}_1}{dt} = \omega_2{}^{-1}\frac{d\mathcal{E}_2}{dt} = -\omega_3{}^{-1}\frac{d\mathcal{E}_3}{dt}.$$

Relation (5), however, gives deeper insight into the foundations of the symmetry of U, and also simplifies its derivation greatly. If we devise a canonical transformation to satisfy (again neglecting field-particle resonances[16])

$$K = H_0 + \langle K^{(2)}\rangle + K^{(3)} + O(\epsilon^4),$$

then the static component of (5) yields the formula

$$-c^{-1}\int d^3x \langle \vec{J}^{(2)} \cdot \vec{A}\rangle = 3\int d\Gamma\, f_0 \langle K^{(3)}\rangle,$$

where $K^{(3)}$ represents the single-particle trilinear interaction energy. We thus are led to an interpretation of the three-wave coupling coefficient as the trilinear interaction energy of a single particle in the fields of the three modes, summed over all nonresonant particles; the symmetry is a necessary consequence of the trilinearity. This interpretation is, of course, consistent with that in terms of the trilinear interaction *Lagrangian* which one requires from the viewpoint of the averaged-Lagrangian method.[5] The great advantage of our Hamiltonian formulation is that Dewar's operator formalism provides us immediately with a formula for $K^{(3)}$, viz.,

$$K^{(3)} = H^{(3)} = \{H^{(2)}, W^{(1)}\} + 3^{-1}\{\{H^{(1)}, W^{(1)}\}, W^{(1)}\},$$

where the generating function $W^{(1)}$ is found by solving

$$\partial_t W^{(1)} + \{W^{(1)}, H_0\} = H^{(1)}.$$

The integral of $K^{(3)}$ over phase space reproduces the Poisson-bracket formula for the coupling coefficient,[7,17] previously derived by the method of generalized ponderomotive forces.

This work was supported by the U. S. Department of Energy under contracts No. W-7405-ENG-48 and No EY-76-S-02-2456.

[1] *Advances in Plasma Physics*, edited by A. Simon and W. B. Thompson (Interscience, New York, 1976), Vol. 6, pt. 1.

[2] R. C. Davidson, *Methods in Nonlinear Plasma Theory* (Academic, New York, 1972); V. N. Tsytovich, *Theory of Turbulent Plasma* (Plenum, New York, 1977).

[3] L. Altshul and V. Karpman, Zh. Eksp. Teor. Fiz. 47, 1552 (1964) [Sov. Phys. JETP 20, 1043 (1965)].

[4] E. G. Harris, in *Advances in Plasma Physics*, edited by A. Simon and W. B. Thompson (Interscience, New York, 1970), Vol. 3.

[5] J. P. Dougherty, J. Plasma Phys. 4, 761 (1970); J. J. Galloway and H. Kim, J. Plasma Phys. 6, 53 (1971).

[6] S. Johnston, Phys. Fluids 19, 93 (1976).

[7] S. Johnston and A. N. Kaufman, in *Plasma Physics*, edited by H. Wilhelmsson (Plenum, New York, 1977),

p. 159.
[8]S. Johnston, A. N. Kaufman, and G. L. Johnston, Lawrence Berkeley Laboratory Report No. LBL-7251, 1978 (to be published).
[9]R. L. Dewar, Phys. Fluids 16, 1102 (1973).
[10]A survey of perturbation methods for Hamiltonian systems has been given by G. E. O. Giacaglia, *Perturbation Methods in Non-Linear Systems* (Springer-Verlag, Berlin, 1972). The author notes (p. 144) that "Lie transform techniques are quite popular at present and they actually represent a real breakthrough from classical methods. At least one can say they were not known to Poincaré, a hard thing to discover in perturbation theories. The credit for this new method goes to Hori. Later works and modified algorithms should only be considered as refinements or different forms of the same basic idea."
[11]G. Hori, Publ. Astron. Soc. Jpn. 18, 287 (1966); A. Deprit, Celestial Mech. 1, 12 (1969).
[12]H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, Mass., 1950), Sect. 8-1.
[13]R. L. Dewar, J. Phys. A 9, 2043 (1976).
[14]L. D. Landau and E. M. Lifshitz, *Mechanics* (Addison-Wesley, Reading, Mass., 1960), Sect. 40.
[15]J. R. Cary and A. N. Kaufman, Phys. Rev. Lett. 39, 402 (1977).
[16]In the case of a field-particle resonance, the perturbation procedure breaks down and a two-time-scale refinement of the transformation becomes necessary. See Ref. 9.
[17]S. Johnston and A. N. Kaufman, Lawrence Berkeley Laboratory Report No. LBL-7252, 1978 (to be published).
[18]P. A. Sturrock, Ann. Phys. (N.Y.) 9, 422 (1960).

Boojums in Superfluid ^{3}He-*A* and Cholesteric Liquid Crystals

D. L. Stein, R. D. Pisarski, and P. W. Anderson[(a)]
Department of Physics, Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540
(Received 26 January 1978)

Because of the similarity of their order parameters, there are close analogies between defects of ^{3}He-*A* and cholesteric liquid crystals. In particular, boojums, originally predicted for ^{3}He-*A*, should exist as well in cholesterics. Certain textures experimentally observed and reported in the literature are identified as boojums. A topological analysis is given, and the effects of boojums on dynamical properties of cholesterics are discussed.

It is known that the topological characteristics of the boundary of a system with spontaneously broken symmetry are connected to the appearance of singularities in the order parameter. It follows that in order to fully treat the problem of textures in the bulk, one must consider global boundary conditions as well as the topology of the order parameter. This approach has been of great interest recently in determing the properties of ^{3}He-*A* in containers of various topologies.[1-3] One of the most interesting conclusions drawn from these studies is the necessity of new types of surface singularities in the order parameter of ^{3}He-*A* in certain containers. For example, if ^{3}He-*A* is placed in a sphere subject to the boundary condition that the two gap-parameter vectors $\vec{\Delta}_1$ and $i\vec{\Delta}_2$ are constrained to lie in the surface, a singularity in the pair angular momentum vector $\vec{l}$ must appear, and it is believed that the lowest-energy configuration consists of an isolated singularity lying on the surface of the sphere. This type of singularity is known as a boojum[3] (see Fig. 1), and connects to a nonsingular vortex texture in the bulk. The appearance of a boojum in a sample of ^{3}He-*A* can lead to decay of superflow without the nucleation of highly singular vortex-line cores.[3,4] It there-

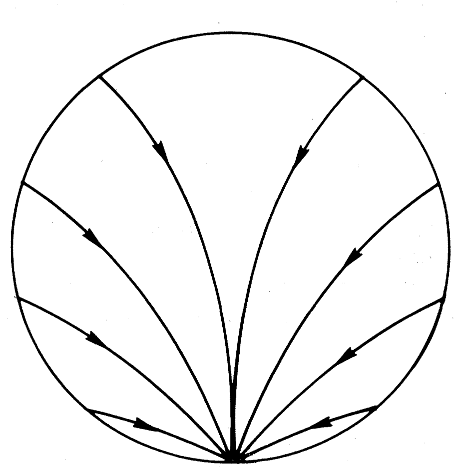

FIG. 1. A simple spherical boojum. Lines represent $\vec{l}$ in ^{3}He-*A*, $\vec{t}$ in cholesterics.

