y, is given exactly by

$$x_j(y) = \left[\frac{(1+y)(C_j+\eta_j y)(A_j - B_j y)}{1+3\eta_j(A_j - B_j y)}\right]^{1/2}. \qquad (15)$$

Here we have $A_j = 1 - \delta_j + \eta_j$, $B_j = 1 + \delta_j + 2\eta_j$, and $C_j = 1 + 3\delta_j + \eta_j$, where $\delta_\tau = 0$, $\eta_\tau = Q_\tau / 6m_\pi$, $\delta_{\tau'} = (m_\pi - m_{\pi^0})/3m_{\pi^0}$, $\eta_{\tau'} = Q_{\tau'}/6m_{\pi^0}$.

It follows from Eq. (14) that the distribution in unlike and like pion energies will be $1 + a_j T_3/m_K$ and $1 + \gamma_j T_1/m_K$, respectively [except for a phase-space factor similar to Eq. (15)], and that we have

$$a_j = \frac{3\beta_j}{1 - \beta_j(Q_j/m_K)}, \qquad (16)$$

and

$$\gamma_j = -\tfrac{1}{2}a_j. \qquad (17)$$

In a recent study of 959 τ events,[6] it was observed that the deviations from "phase space" of the distributions in T_3 and in T_1 were roughly linear, with $a_\tau = 6.8 \pm 1.2$, $\gamma_\tau = -2.2 \pm 0.3$, in fair agreement with Eq. (17) and hence with our neglect of quadratic terms in Eq. (11). We then have $\beta_\tau \sim 1.3$, in agreement with our expectation that the $a_j(n,l)$ should be of comparable magnitude. We predict $\beta_{\tau'} \sim -2.6$. The intrinsic uncertainty in this prediction, due to neglect of quadratic terms in Eq. (11), is probably about 20%. By careful study of existing τ data, it should be possible to estimate $a_\tau(2,0)$ and $a_\tau(2,2)$ [for example by checking (17) more accurately] and hence refine the accuracy of our prediction of the π^+ energy distribution in τ' decay.[7]

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[5]This remark applies only to effects arising within the range of the strong interactions. In order to take account of the long-range Coulomb final-state interaction in τ decay it is necessary to multiply the right-hand side of (8) by a Coulomb correction factor $1 + (\alpha\pi/2)[(1/v_{13}) + (1/v_{23}) - (1/v_{12})]$, as shown by Dalitz, reference 1. Empirical distributions should be divided by the square of this factor before analyzing to find the $a_\tau(n,l)$ coefficients. This was not done here, and the values quoted below for β are therefore too small. We wish to thank R. H. Dalitz, H. P. Noyes, and M. A. Ruderman for helpful discussions on this point.

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[7]R. H. Dalitz has performed an analysis of 900 earlier τ events [Reports on Progress in Physics (The Physical Society, London, 1957), Vol. 20; and private communication] and obtains $\beta_\tau = 1.6 \pm 0.5$ and $\mathrm{Re}\{[a_\tau(2,0) - a_\tau(2,2)]/a_\tau(0,0)\} = -8.4 \pm 6.5$.

HIGH-ENERGY LIMIT OF SCATTERING CROSS SECTIONS

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Pomeranchuk[1] has shown that under rather plausible assumptions concerning the very high energy dependence of total cross section—the main one being that they behave almost as constants at infinity—the difference of the cross sections for a particle and its charge conjugate on the same target tends to vanish at infinity. More explicitly: if $\sigma^+(E)$ and $\sigma^-(E)$ are the total cross sections for a particle and its charge conjugate ($\pi^+ - \pi^-$, proton-antiproton, $K^+ - K^-$, $K^0 - \bar{K}^0$, etc.) at total energy E of the incoming particle in the laboratory system on a specific target, and if

$$\lim_{E\to\infty} \sigma^+(E) = \sigma^+(\infty),$$

$$\lim_{E\to\infty} \sigma^-(E) = \sigma^-(\infty), \qquad (1)$$

$\sigma^+(\infty)$ and $\sigma^-(\infty)$ being finite constants (or zero), then Pomeranchuk shows that

$$\sigma^{+}(\infty)=\sigma^{-}(\infty). \tag{2}$$

Pomeranchuk tries to justity (1) by pointing out that due to the finite range of interaction between particles, the quantity

$$A^{\pm}(E)/E,$$

where $A^{\pm}(E)$ is the forward scattering amplitude for the corresponding process averaged over possible spins, should go to a limit with increasing E. Though it seems evident that the said quantity is bounded, it is not clear that if it does not go to zero it tends to a limit. But for Pomeranchuk's argument (1) it is essential that at least the imaginary part has a limit. We shall explore some consequences that can be drawn if the condition

$$\lim_{E\to\infty} A^{\pm}(E)/E=\text{constant} \tag{3}$$

really is fulfilled. We note that by virtue of the optical theorem the condition (3) on the imaginary part of $A^{\pm}(E)$ is equivalent to (1); for the real part, $D^{\pm}(E)$, (3) implies

$$\lim_{E\to\infty}\frac{D^{\pm}(E)}{E}=\text{constant}. \tag{4}$$

The property (2) is of importance for the application of dispersion relations: in fact, in many cases where the unsubtracted dispersion relations for the difference of particle and antiparticle amplitudes can converge because of the vanishing of $\sigma^{+}-\sigma^{-}$ at very high energies. Such dispersion relations, subject to that hope, were used and analyzed for several processes and proved to give meaningful results. This is the case for π-N S-wave scattering[2] (for the combination $\alpha_1-\alpha_3$) and for K-N scattering.[3] In order that those applications make any sense, it is necessary and sufficient, however, that integrals of the type

$$\int^{t}\frac{\sigma^{-}(E)-\sigma^{+}(E)}{E}dE \tag{5}$$

converge for $t\to\infty$. Even if (2) makes more plausible the convergence of (5), it is by no means sufficient: to ensure it, it would be necessary to know how $\sigma^{-}-\sigma^{+}$ goes to zero at very high energies.

We note that if

$$\lim_{E\to\infty}[\sigma^{-}(E)-\sigma^{+}(E)]\to(1/\ln E) \tag{6}$$

(behavior which in some cases has been interpreted to be Pomeranchuk's prediction), then (5) would diverge as $\ln\ln t$. In reference 1, however, no discussion is made on how the limit (2) is reached.

We want to show that using the same starting point as Pomeranchuk, something can be said on how $\sigma^{-}-\sigma^{+}$ goes to 0; in fact we show in this note that the integral (5) is indeed convergent.

Let us start from the subtracted dispersion relation whose convergence is guaranteed by (1):

$$D^{+}(E)=\tfrac{1}{2}\left(1+\frac{E}{M}\right)D^{+}(M)+\tfrac{1}{2}\left(1-\frac{E}{M}\right)D^{-}(M)+\sum_b\frac{A_b p^2}{E+E_b}+\frac{p^2}{\pi}\int_{E_0}^{\infty}\left(\frac{\sigma^{+}(E')}{E'-E}+\frac{\sigma^{-}(E')}{E'+E}\right)\frac{dE'}{p'}, \tag{7}$$

where $p=(E^2+M^2)^{1/2}$, M being the mass of the particle in question and $\sum_b A_b p^2/(E+E_b)$ the contribution of possible poles (bound states at energies E_b), A_b being constants. The lowest limit of integration E_0 depends on the process in question (possible existence of unphysical regions). We could start from the dispersion relation for $D^{-}(E)$ without changing the conclusions we shall reach.

Let us define

$$f(E)=\sigma^{-}(E)-\sigma^{+}(E); \tag{8}$$

then (7) can be rewritten

$$\frac{D^{+}(E)}{E}=\tfrac{1}{2}\left(\frac{1}{E}+\frac{1}{M}\right)D^{+}(M)+\tfrac{1}{2}\left(\frac{1}{E}-\frac{1}{M}\right)D^{-}(M)+\sum_b\frac{A_b(E-M^2/E)}{E+E_b}$$
$$+\left(E-\frac{M^2}{E}\right)\frac{2}{\pi}\int_{E_0}^{\infty}\frac{E'\sigma^{+}(E')}{p'(E'^2-E^2)}dE'+\left(E-\frac{M^2}{E}\right)\frac{1}{\pi}\int_{E_0}^{\infty}\frac{f(E')}{E'(E'+E)}dE'. \tag{9}$$

The condition (4) is clearly satisfied by the terms containing $D^{\pm}(M)$ and the bound-state contributions on the right-hand side of (9). This is also the case for the integral over σ^+: if in fact we subdivide the integration at an energy ϵ sufficiently high so that for $E'>\epsilon$, $\sigma^+(E')$ can be considered a constant, then its contribution for $E \gg \epsilon$ is essentially given by

$$\frac{2}{\pi}\frac{1}{E}\int_{E_0}^{\epsilon}\frac{E'}{p'}\sigma^+(E')dE' - \frac{2}{\pi}\frac{\epsilon}{E}\sigma^+(\infty). \qquad (10)$$

Then, if we call

$$G(E) = \int_{E_0}^{\infty}\frac{\varphi(E')dE'}{(E'+E)}, \text{ with } \varphi(E) = f(E)/p, \qquad (11)$$

condition (4) implies that

$$\lim_{E\to\infty} EG(E) = \text{constant}. \qquad (12)$$

On the other hand, the following theorem on Stieltjes transforms holds[4]:

Theorem: Let $G(E)$ be the (convergent) integral defined by (11) $(E_0 > 0)$ and let

$$\lim_{E\to\infty}\frac{1}{E}\int_{E_0}^{E} E'\varphi(E')dE' = 0; \qquad (13)$$

then

$$\lim_{E\to\infty}\left(EG(E) - \int_{E_0}^{E}\varphi(E')dE'\right) = 0. \qquad (14)$$

The condition (13) is trivially satisfied in our case because of (2) [although it is less restrictive than (2) since it implies only the existence of limits (1) and (2) in the mean]. From (14) and (12), the convergence of the integral

$$\int_{E_0}^{\infty}\varphi(E')dE' = \int_{E_0}^{\infty}\frac{\sigma^-(E') - \sigma^+(E')}{p'}dE' \qquad (15)$$

follows, as was to be demonstrated.[5]

As a consequence, we see that the limiting behavior (6) is by no means compatible with the constancy of cross sections at high energies, i.e., the difference $\sigma^-(E) - \sigma^+(E)$ must go to zero faster than $1/\ln E$.

We want to note, finally, that an immediate consequence of our conclusion is that

$$\lim_{E\to\infty}\Delta(E) = 0, \qquad (16)$$

where

$$\Delta(E) = [D^-(E) - D^+(E)]/E. \qquad (17)$$

By using the unsubtracted dispersion relation for $D^-(E) - D^+(E)$ whose validity we have just discussed, we can write

$$\Delta(E) = \sum_b \frac{B_b}{E^2 - E_b{}^2} + \frac{2}{\pi}\int_{E_0}^{\infty}\frac{p'f(E')}{E'^2 - E^2}dE', \qquad (18)$$

where B_b are the constant residua of the bound-state contributions. Changing the integration variable to E'^2, the integral of the right-hand side of (18) is reduced to the form of a Hilbert transform. Its convergence and the possibility of inverting the Hilbert transform[6] are ensured by the existence of (15). This means that the expression

$$\int^{\infty}\frac{\Delta(E')}{E'^2 - E^2}dE'^2$$

must converge for all values of E (and go to zero for $E\to\infty$), from which (16) follows.

An empirical discussion on the limit value for $\Delta(E)$ was given by Goldberger et al.[2] in connection with the possibility of using (18) as a sum rule. Here instead we have shown, from rather general theoretical ground, that (18) is valid without subtraction constants and that, as a consequence, the limit (16) holds.

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[2] Goldberger, Miyazawa, and Oehme, Phys. Rev. 99, 986 (1955). See also: A. Salam, Proceedings of the CERN Symposium on High-Energy Accelerators and Pion Physics, Geneva, 1956 (European Organization of Nuclear Research, Geneva, 1956). Vol. II, p. 179.

[3] P. T. Matthews and A. Salam, Phys. Rev. 110, 569 (1958); R. H. Dalitz, 1958 Annual International Conference on High-Energy Physics at Cern, edited by B. Ferretti (CERN Scientific Information Service, Geneva, 1958), p. 192; E. Galzenati and B. Vitale, Phys. Rev. 113, 1635 (1959); D. Amati, Phys. Rev. 113, 1692 (1959).

[4] This is a slight modification of a thorem by G. H. Hardy and J. E. Littlewood, Proc. London Math. Soc. 30, 34 (1930). Compare also with D. V. Widder, The Laplace Transform (Princeton University Press, 1946), Lemma 5, p. 193.

[5] (14) shows that if $D^{\pm}(E)/E$ satisfies only the weaker condition to be bounded at infinity, then the existence of the integral (5) does not necessarily follow. The following example: $\sigma^- - \sigma^+ \sim \sin(\ln E)/\ln E$ ($\to 0$ for $E\to\infty$), for which $EG(E) \sim \sin(\ln E)$, gives an oscillating behavior of $\int_{E_0}^{E}\varphi(E')dE'$ for $E\to\infty$. On the other hand, if the limit (4) exists, then the integral (15) converges also in the more general case of bounded σ^+ and σ^- without the requirement of the actual existence of a limit for $\sigma^- - \sigma^+$ at $E\to\infty$. This last fact is the consequence of another theorem of Hardy and Littlewood (reference 4). A trivial example: $\sigma^- - \sigma^+ \sim \sin E$.

[6] E. C. Titchmarsh, Introduction to the theory of Fourier Integrals (Oxford Press, 1948), p. 724.