[4]D. A. Bromley, E. Almqvist, H. E. Gove, A. E. Litherland, E. B. Paul, and A. J. Ferguson, Phys. Rev. 105, 957 (1957).

[5]R. O. Bondelid, J. W. Butler, C. A. Kennedy, and A. del Callar (to be published) (private communication from R. O. Bondelid).

[6]It is of considerable importance that the discrepancy between the $C^{12}(He^3, n)O^{14}$ threshold measurements of references 4 and 5 be resolved. In the present paper, we will use the $O^{14}(\beta^+)N^{14*}$ end-point kinetic energy of 1800 ± 6.5 kev calculated from the threshold value of reference 5.

[7]J. Fischer, B. Leontic, A. Lundby, R. Meunier, and J. P. Stroot, Phys. Rev. Letters 3, 349 (1959), $\tau_\mu = 2.20 \pm 0.015$ μsec. R. A. Reiter, T. A. Romanski, and R. B. Sutton, post-deadline paper, Washington APS Meeting, 1960, $\tau_\mu = 2.211 \pm 0.003$ μsec. V. L. Telegdi, R. A. Swanson, R. A. Lundby, and D. D. Yovanovitch (private communication from Dr. Yovanovitch), $\tau_\mu = 2.208 \pm 0.004$ μsec.

[8]We are indebted to Professor R. H. Dalitz for pointing out to us that the results of the perturbation theory calculations for the muon are ambiguous to the extent that one is at liberty to regularize in different ways the muon and electron self-energy graphs and the vertex graphs, thereby adding finite constant terms to the decay amplitude. The choice of reference 2 corresponds to regularizing only the photon propagators.

[9]To lowest order in α, a consistent treatment of the decay accompanied by inner bremsstrahlung yields results identical with those of perturbation theory, with the usual weak-coupling constant G replaced by $\sqrt{2}\,a(s_0, \lambda)$.

[10]The factor $\sqrt{2}$ is introduced to conform to the notation for G of reference 1.

[11]M. Morita, Phys. Rev. 113, 1584 (1959).

[12]The failure to obtain a finite result for $A(s)$ when the electromagnetic form factor is included in the calculation of $\mathrm{Im}A(s)$ contradicts previous conjectures (see the first two papers of reference 2). We remark, however, that the form factor is necessary and sufficient to secure convergence in the integral for $B(s)$ and in the contributions to the neutron decay amplitude involving interactions of the electron with the anomalous magnetic moments of the nucleons. These form-factor-dependent terms do not contribute significantly to the corresponding decay rates.

[13]A term in the total electromagnetic correction of the form $-Z\alpha\pi/v_e$ is identified in the usual manner with the term of order $Z\alpha$ in the expansion of the Fermi factor for the positron decay, and is omitted from the corrections designated as radiative.

SOME APPLICATIONS OF THE GENERALIZED UNITARITY RELATION

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Recent studies of the analytic continuations of perturbation theory amplitudes have led to a greatly increased understanding of their properties.[1-10] Landau[4] showed the existence of a class of singularities which are a generalization of the ordinary thresholds for the onset of new physical processes. He derived two conditions for determining the location of these singularities: (1) Each of the intermediate particles (which correspond to lines joining otherwise disjunct subgraphs of the original Feynman graph) is on the mass shell. (2) The four-momenta of these intermediate particles satisfy a geometrical relation which generalizes the requirement, for the ordinary thresholds, that the intermediate particles all be at rest in the center-of-mass frame.

Applications of dispersion relations have all made use of the fact that the discontinuity across a branch cut starting from an ordinary physical threshold is obtained from the unitarity condition. It has been shown, for all finite order graphs, that a generalized unitarity relation gives the discontinuity across a branch cut which starts from any Landau singularity.[11] In this note we point out, by means of some simple examples, that this generalized unitarity relation greatly augments the power of analytic continuation methods.

The rule for calculating the discontinuity of a graph is that in the integral over the virtual four-momenta, for each line of the "reduced graph" which determines the singularity the factor $(M^2 - q^2)^{-1}$ is replaced by $2\pi i\delta(q^2 - M^2)$. In other words, the subgraphs which comprise the vertices of the reduced graph are considered only for the case that the lines leading into them represent free particles. The products of the factors $\delta(q^2 - M^2)$ and d^4k for the reduced graph are analogous to phase space volume elements.

Certain reduced graphs lead to poles rather than branch points; in these cases the residue is obtained by considering the lines to be on the mass shell. The poles discussed by Chew and Low[12] are simple examples.

The Landau singularities are, in fact, all the singularities of a finite order graph, apart from trivial exceptional branch points that may occur on secondary Riemann sheets.[11] Since the discontinuities and residues of a given graph are expressible in terms of lower order graphs, the generalized unitarity relation can be used to generate the perturbation series expansion, when it is supplemented by appropriate assumptions about the number of necessary subtractions.[13] Therefore, the generalized unitarity relation may be considered to embody the entire content of a convergent perturbation expansion.

As a first illustration, we consider the anomalous threshold of the reaction $\pi+\pi\rightarrow\Sigma+\Sigma$, which has already been discussed by Mandelstam.[14] All graphs which possess the anomalous threshold in question have the structure shown in Fig. 1(a). The circles stand for any graphs which contribute to the $(\pi\Sigma\Lambda)$ vertex, or to π - π scattering. When we sum over all graphs which contribute to the discontinuity above the anomalous threshold, these become the renormalized coupling constant and the elastic scattering amplitude. We follow Mandelstam and treat the particles as spinless and chargeless; the discontinuity of $T_{\pi\Sigma}(E, \cos\theta)$ is then

$$[T_{\pi\Sigma}(E,\cos\theta)] = (2\pi^{-1})g^2\int d^4k\, T_{\pi\pi}(E, \cos\theta') \times \delta(q_+^{\,2}-M_\pi^{\,2})\delta(q_-^{\,2}-M_\pi^{\,2})\delta(q_0^{\,2}-M_\Lambda^{\,2}), \quad (1)$$

where $q_\pm=\frac{1}{2}P\pm k$ and $q_0=p_\Sigma-k$ (we denote by P the total four-momentum, and by $2p_\Sigma$ the difference of the four-momenta of the Σ's). If we expand in partial waves, we obtain

$$[T_{\pi\Sigma}^{\;\,l}(E)] = g^2(8p_\Sigma E)^{-1}P_l(x)\,T_{\pi\pi}^{\;\,l}(E). \quad (2)$$

Here x, the cosine of the angle between the Σ and the intermediate π, is given by $M_\Lambda^{\,2}+p_\pi^{\,2}+p_\Sigma^{\,2} = 2p_\pi p_\Sigma x$. At the anomalous threshold, $x=1$; as E increases to $2M_\pi$, $x\rightarrow+\infty$. Equation (2) for $l=0$ corresponds to Mandelstam's result, but it is here shown to include the contribution of all orders of perturbation theory.

The electromagnetic form factor of the deuteron has a first threshold corresponding to the reduced diagram 1(b).[1] When all graphs are summed, the circles correspond to the asymptotic Bethe-Salpeter amplitude and to the nucleon form factor. Integration leads to the usual zero-range potential result, with some relativistic corrections. The next threshold corresponds to graph 1(c), which has an intermediate pion. The pion production vertex has two nucleon pole terms, which appear to be dominant. It follows from a theorem proved in reference 11 that in calculating the discontinuity associated with the reduced graph 1(c), these poles never lie in the region of integration, but they are actually quite close to the edge. The pole terms can be interpreted as the lowest order correction to the wave function arising from the finite range of the force. The remainder of the pion production vertex gives a "nonadditivity" correction.

The technique described above can also be used to calculate, and to give a physical interpretation to, the Mandelstam spectral functions. These correspond to a division of a graph into four (or more) disjunct parts, with each external line being attached to a separate part. Consider first the reduced nucleon-nucleon scattering graph 2(a). The corresponding Mandelstam spectral function is just the renormalized fourth order perturbation theory result.[2,3]

Another reduced two-meson exchange graph is shown in Fig. 2(b). When all contributing Feyn-

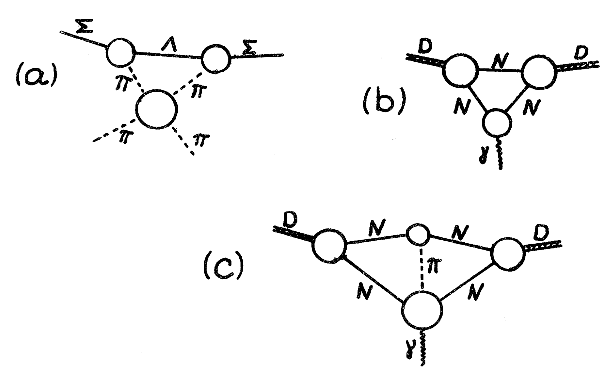

FIG. 1. Reduced graphs corresponding to some "anomalous" thresholds.

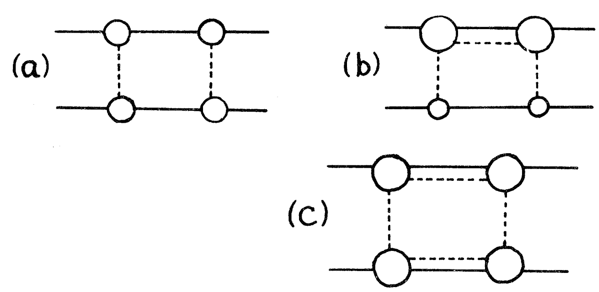

FIG. 2. Reduced graphs corresponding to some Mandelstam spectral thresholds for nucleon-nucleon scattering. The solid lines represent nucleons, and the dashed lines represent pions.

man graphs are summed, one of the circles becomes the meson-nucleon scattering amplitude; by paying careful attention to the branch onto which the amplitude is continued, it can be seen that the other upper circle is the conjugate amplitude. Hence we find, for 2(b),

$$\rho(s,t)=(2\pi i)^{-3}\int d^4k_1\,d^4k_2\,g^2(\gamma\cdot P-M)\delta_1\ldots\delta_5 \times T_{\pi N}^{\dagger}(s',t')T_{\pi N}(s',t''), \tag{3}$$

where $\delta_1\ldots\delta_5$ are the delta functions associated with the five lines of the reduced graph, P is the momentum of the intermediate nucleon in the lower part of the graph, and $s'=(q_4+q_5)^2$ is the energy variable of the meson-nucleon scattering. Let us introduce a dummy integration of the form $\int\delta(s'-(q_4+q_5)^2)ds'$. For s' below the meson production threshold, we find by analytically continuing the ordinary unitarity relation, that

$$(2\pi i)^{-2}\int d^4k_2\,\delta_4\delta_5 T_{\pi N}^{\dagger}(s',t')T_{\pi N}(s',t'') = 2i\,\mathrm{Im}T_{\pi N}(s',t). \tag{4}$$

Hence

$$\rho(s,t)=\pi^{-1}\int ds'\int d^4k_1\,\delta_1\ldots\delta_3\delta(s'-(q_4+q_5)^2) \times g^2(\gamma\cdot P-M)\,\mathrm{Im}T_{\pi N}(s',t). \tag{5}$$

The integration over d^4k_1 is now identical to that which occurs in fourth order perturbation theory,[11] and leads to

$$\rho(s,t)=g^2\int_{s_0}^{s_1} ds'(\gamma\cdot P-M)\,\mathrm{Im}T(s',t)/V(s,t,s'). \tag{6}$$

The denominator V is proportional to the volume of a certain four-dimensional figure. The s' integration goes between the limits $s_0=(M_N+M_\pi)^2$ and a maximum s_1 determined by the equation $V(s,t,s_1)=0$.

If we consider also the reduced graphs with two or more mesons (or other particles) passing between the upper circles in graph 2(b), we find, by using the unitarity condition for the inelastic scattering situation, that these graphs give such a contribution that Eq. (6) is also correct for $s'>(M_N+2M_\pi)^2$. Graph 2(c) may be analyzed by the same method and leads to a result similar to (6), except that the imaginary part of a meson scattering matrix is associated with each nucleon. The crossed reduced graphs may be obtained from crossing symmetry.

We see that the contribution of all two-meson exchange graphs to the Mandelstam spectral functions can be written down in terms of the meson-nucleon scattering amplitude. It is clear, in particular, that meson-meson interactions do not contribute to nucleon-nucleon scattering and to meson-nucleon scattering in fundamentally different ways. Of course, it is unlikely that $\mathrm{Im}T_{\pi N}(s',t)$ in the spectral region can be obtained from the experimental values by a model-independent extrapolation.

It should be noted that there are some Feynman graphs, involving the $\lambda\phi^4$ term in the Lagrangian, which do not contribute to the spectral functions. If a subtracted form of the Mandelstam representation is used, these terms are absorbed.

The cases (such as Σ - Σ scattering) in which the Mandelstam representation is invalid are not greatly different from nucleon-nucleon scattering; there are spectral functions determinable in terms of simpler processes, but they must be integrated over complex surfaces.

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