a consequence of (a) and the $\Delta I = \frac{1}{2}$ rule, the decay rates for the reactions (1), (2), and (3) are approximately equal. (c) The Σ^- will undergo leptonic decay and the Σ^+ will not undergo leptonic decay (with nucleon emission) if one assumes that the weak interactions go only by way of four-fermion couplings. The theory predicts that the sign of the asymmetry parameter for the decay mode (2) is the same as that of the asymmetry parameter in Λ decay, and that the two asymmetry parameters are comparable in magnitude. For odd relative Σ - Λ parity, the present approximate treatment indicates that the decay mode (1) involves a pure S state and the decay mode (3) a pure P state; a measurement of the recoil neutron polarizations could check these consequences of the theory. The asymmetry parameter for the decay mode (3) may be nonzero if the three decay rates are not precisely equal.

If the Σ is a bound $P_{1/2}$ state of Λ^0 and a pion, then the only difference would be that the $\Sigma^+ \rightarrow \pi^+ + n$ decay would be all in P state rather than S state.

One of the authors (S. B.) wishes to thank Professors Sam Schweber and Kenneth Ford for their comments. The other would like to extend similar thanks to Professor G. Feinberg and Professor T. D. Lee.

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[2]F. Eisler et al., Nuovo cimento 7, 222 (1958).

[3]1958 Annual International Conference on High-Energy Physics at CERN, edited by B. Ferretti (CERN Scientific Information Service, Geneva, 1958), p. 268.

[4]It is possible that the Σ^- beta decay will be suppressed relative to the Λ beta decay by the tendency of a four-fermion V-A interaction, which may mediate the virtual Λ beta decay within the Σ^-, to leave the proton with little recoil momentum, and hence largely in a S state relative to the pion within the Σ^-. This would diminish the absorption amplitude estimated crudely in eq. (7) by B'.

[5]P. Nordin et al., Phys. Rev. Letters 1, 380 (1958).

[6]R. L. Cool, B. Cork, J. W. Cronin, and W. A. Wenzel, Phys. Rev. 114, 912 (1959); P. Franzini et al., Bull. Am. Phys. Soc. 5, 224 (1960). The product of asymmetry parameter and average hyperon polarization is 0.02 ± 0.08 for decay mode (1), 0.62 ± 0.12 for decay mode (2), and 0.1 ± 0.17 for decay mode (3). The Σ^- and Σ^+ are produced in $\pi^- n$ and $\pi^+ p$ collisions, respectively, and are known to have a nonzero polarization owing to the substantial up-down asymmetry observed in the decay mode (2). The measurements on the Σ^+ are from Cool, Cork, Cronin, and Wenzel.

ELECTROMAGNETIC CORRECTIONS TO THE DECAYS OF THE MUON, O^{14}, AND THE NEUTRON*

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According to the conserved vector current hypothesis of Feynman and Gell-Mann[1] for the weak interactions, the vector coupling constant G_V should be the same for μ decay and nuclear β decay. However, the values of G_V determined from the experimental muon lifetime and the decay rate in the $0^+ \rightarrow 0^+$ transition $O^{14}(\beta^+)N^{14*}$ differ by a few percent after inclusion of the radiative corrections discussed by a number of authors.[2] Kinoshita and Sirlin predict a muon lifetime of $\tau_\mu = (2.31 \pm 0.05) \times 10^{-6}$ sec on the basis of an O^{14} half-life of 72.5 ± 0.5 sec,[3] and a β^+ end-point kinetic energy of 1810 ± 8 kev.[4] With a recent revised end-point energy of 1800 ± 6.5 kev,[5] the predicted lifetime becomes $\tau_\mu = (2.26 \pm 0.05) \times 10^{-6}$ sec.[6] These numbers are to be compared with the observed lifetime $\tau_\mu = (2.21 \pm 0.005) \times 10^{-6}$ sec.[7] While the discrepancy may not be serious, it should be noted that the calculated radiative corrections are subject to certain ambiguities: the perturbation theory approach used in the calculations of reference 2 yields finite results in the muon case,[8] but the calculated corrections for nuclear β decay are ultraviolet divergent. The cutoff-dependent term in the correction to the decay rate, $\Delta\Gamma/\Gamma_0 = (3\alpha/2\pi)\ln(\Lambda^2/m_p m_e)$, was evaluated by choosing a cutoff Λ equal to the proton mass m_p, in the expectation that the proper inclusion of nuclear electromagnetic form factors would introduce a natural cutoff of the virtual photon contributions near this value. The resulting expression $\Delta\Gamma/\Gamma_0 = (3\alpha/2\pi)\ln(m_p/m_e) = 0.026$ represents the dominant part of the quoted radiative correction. This rather unsatisfactory situation makes it desirable to re-examine the problem from a somewhat different point of view. In the present paper, we wish to discuss the results of a calculation based on the techniques of dispersion theory.

The S-matrix element for the process $\mu \rightarrow e + \nu + \bar{\nu}$ may be written

$$\langle e\nu\bar{\nu}|S|\mu\rangle = i(2\pi)^4\delta(\mu - e - \nu - \bar{\nu})(16e_0\mu_0\nu_0\bar{\nu}_0)^{-1/2}F(e\nu\bar{\nu};\mu), \quad (1)$$

where $\mu, e, \nu, \bar{\nu}$ denote the four-momenta of the particles. From invariance under proper Lorentz transformations, it follows that F has the general structure

$$F = (\bar{u}(\nu)\gamma_\alpha(1+\gamma_5)v(\bar{\nu}))\bar{u}(e)\{\gamma_\alpha[a(1+\gamma_5) + b(1-\gamma_5)] - i(\mu+e)_\alpha[c(1+\gamma_5)+d(1-\gamma_5)]\}u(\mu), \quad (2)$$

provided the usual two-component neutrino theory is assumed. For convenience, the free particle spinors are normalized according to

$$\bar{u}(p)\gamma_\alpha u(p) = 2ip_\alpha. \quad (3)$$

To lowest order in the weak interaction, the form factors, a, b, c, and d are functions of the single invariant parameter

$$s = -(\mu - e)^2 = -(\nu + \bar{\nu})^2. \quad (4)$$

In accordance with the conventional theory,[1] we require that, in the absence of electromagnetic interactions, $b = c = d = 0$, and $a =$ constant. The assumption of microscopic causality leads, by the usual heuristic methods, to the conclusion that these functions are analytic in the complex s plane, except for singularities on the real axis. In the resulting dispersion relations for these functions, the absorptive parts may be determined from unitarity as a sum over all intermediate states in the scattering process, $\nu + \bar{\nu} \rightarrow \mu + \bar{e}$.

In contrast to the analogous problem in meson theory, all intermediate states of the form $\mu' + \bar{e}' + n\gamma$ produce singularities extending down to the threshold at $s = (m_\mu + m_e)^2$, and none can be neglected on the basis of its remoteness from the region of physical interest. However, the fine structure constant α appears to higher than first power in the contribution from all states save those of the form $\mu' + \bar{e}'$ [Fig. 1(a)]. When only these states are retained in the absorptive part, the dispersion relations become a set of integral equations for the invariant functions, accurate to order α. The amplitude for $\mu - \bar{e}$ scattering appearing in these equations is

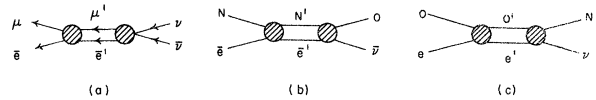

FIG. 1. The lowest order dispersion diagrams encountered in the calculation of the absorptive part of the decay amplitude for (a) the muon, and (b), (c) the $0^+ \rightarrow 0^+$ decay $O^{14} \rightarrow N^{14*}$.

therefore evaluated in Born approximation, and the equations solved by a single iteration, taking for zeroth order the values: $a =$ const, $b = c = d = 0$. If unsubtracted dispersion relations are assumed for b, c, and d, convergent integrals are obtained for these functions. However, the imaginary part of a contains a term

$$(s - m_e^2 - m_\mu^2)(4p^2s)^{-1/2}\int_0^{4p^2}\frac{dq^2}{q^2+\lambda^2}, \quad (5)$$

where $4p^2s = [s - (m_\mu + m_e)^2][s - (m_\mu - m_e)^2]$, and λ is the usual fictitious photon mass introduced to circumvent the infrared problem. For large s, this term behaves as $\ln(s/\lambda^2)$. Consequently, a dispersion integral with a single denominator would not converge, and it is necessary to use a dispersion relation for $a(s)$ with one subtraction,

$$a(s) = a(s_0, \lambda) + \frac{s - s_0}{\pi}\int_{(m_\mu+m_e)^2}^{\infty}\frac{ds'\,\mathrm{Im}a(s')}{(s'-s)(s'-s_0)}, \quad (6)$$

in which there appears a phenomenological constant $a(s_0, \lambda)$.

The dependence of the decay rate on terms which diverge in the limit $\lambda \rightarrow 0$ must be removed when contributions from the bremsstrahlung process $\mu \rightarrow e + \nu + \bar{\nu} + \gamma$ are introduced.[9] With an arbitrary choice of subtraction point s_0, this would require that the constant $a(s_0, \lambda)$ itself contain an infrared divergence. However, explicit cancellation is found to occur if and only if the choice $s_0 = (m_\mu - m_e)^2$ is made. This circumstance makes it highly attractive to define the renormalized coupling constant for μ decay as[10]

$$G_\mu = \sqrt{2}\,a((m_\mu - m_e)^2). \quad (7)$$

Using this definition, we obtain for the electron spectrum results (DLM) differing from those of perturbation theory (KS)[2] only by the term

$$\left(\frac{dN}{dx}\right)_{\rm DLM}-\left(\frac{dN}{dx}\right)_{\rm KS}=\frac{5\alpha}{4\pi}\left(\frac{11}{5}-\ln\frac{m_\mu}{m_e}\right)\left(\frac{dN}{dx}\right)_0, \tag{8}$$

where $(dN/dx)_0$ denotes the statistical spectrum. The resulting electron decay rate is decreased by 1.32% from its value in the absence of electromagnetic corrections. However, to order α, the Michel parameter is unchanged from the result of the perturbation theory calculations,[2] $\rho_{\rm eff}\approx 0.708$.

Similar techniques may be applied to the decay of the neutron and to the $0^+\to 0^+$ transitions $O^{14}(\beta^+)N^{14*}$, $Al^{26*}(\beta^+)Mg^{26}$, and $Cl^{34}(\beta^+)S^{34}$. For $0^+\to 0^+$ transitions, the S-matrix element in covariant form has the structure

$$\langle \bar{e}\nu p_2|S|p_1\rangle = i(2\pi)^4\delta(p_1-p_2-\bar{e}-\nu)(16\bar{e}_0\nu_0p_{10}p_{20})^{-1/2} \times F(\bar{e}\nu p_2;p_1), \tag{9}$$

where $p_{1,2}$ denote the nuclear 4-momenta, and

$$F=i[A(s)(p_1+p_2)_\alpha+B(s)(p_1-p_2)_\alpha] \times(\bar{u}(\nu)\gamma_\alpha v(\bar{e})), \tag{10}$$

In addition to the indicated dependence on the invariant parameter $s=-(p_2+\bar{e})^2$, the functions A and B depend on the momentum transfer $Q^2=(p_1-p_2)^2$. This dependence may be adequately accounted for, however, by the usual multipole expansion of the nuclear matrix elements.[11] We therefore utilize dispersion relations for A and B as functions of s, holding Q^2 fixed at the static nuclear value, $-(M_1-M_2)^2=-(\Delta E)^2$. In the absence of electromagnetic interactions, we require $B(s)=0$ and $A(s)=\text{const}\times M$, where M is the usual nuclear matrix element for the allowed Fermi transition. The contributions which we included in the iterative procedure are represented, for the case of O^{14}, by the dispersion graphs shown in Figs. 1(b) and 1(c). In the amplitude for scattering of the β ray by the nucleus, nuclear electromagnetic form factors may be introduced. The dominant term in the absorptive part appears as a contribution to $\mathrm{Im}A(s)$, and is again of the type given by (5), with appropriate change of the masses, and with an electromagnetic form factor $F(q^2)$ appearing under the integral sign. For large s, this quantity now approaches a constant value, instead of diverging logarithmically, but a single subtraction is still necessary in the dispersion relation for $A(s)$.[12] Explicit cancellation of the infrared-divergent parts now requires that the subtraction point s_0 be chosen to correspond (in the nucleon rest frame, and in the static limit) to the (unphysical) positron energy

$$\bar{e}_0\approx 2m_e[\pi(2Z+1)]^{-1}, \tag{11}$$

where Z is the atomic number of the final nucleus. In analogy with the muon case, we treat the value of A at this point as a phenomenological constant, and define the renormalized coupling constant G_V for the $0^+\to 0^+$ transitions by

$$A(s_0)=(G_V/\sqrt{2})M. \tag{12}$$

The radiative corrections[13] to the spectrum of O^{14} calculated by this means are shown in Fig. 2. In addition to electromagnetic (radiative) effects, we have considered various other corrections to the decay rates which might be expected to be of the same order of magnitude. These corrections are summarized in Table I. Final numerical results for the effective ft values for the transitions considered are also presented in Table I. Apart from numerical factors, the constant G_V is in each case defined to be the value of the subtraction constant appearing in the dispersion relations, with the choice of subtraction point dictated by the requirement of cancellation of

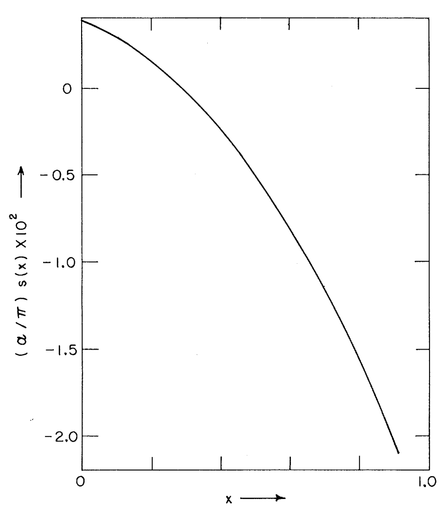

FIG. 2. The radiative electromagnetic correction (see reference 13) to the positron spectrum in the decay of O^{14}. The radiative contribution to the spectrum is $(dN/dx)_{\rm rad}=(\alpha/\pi)s(x)(dN_0/dx)$, where dN_0/dx is the statistical spectrum, not including the Fermi factor.

Table I. Corrections and ft values for nuclear beta decays.

	n	O^{14}	Al^{26*}	Cl^{34}
Corrections(%)				
Electromagnetic (radiative)[a]	-0.56	-1.03	-1.25	-1.62
Nuclear electromagnetic form factors	-0.01	+0.106	+0.328	+0.639
Competition from K capture	...	0.090	0.078	0.068
Electron screening	...	0.093	0.113	0.127
"Second forbidden" nuclear matrix elements[b]	...	-0.024	-0.068	-0.124
Coulomb corrections to the nuclear matrix element[c]	...	-0.35	+0.40 ± 0.2	1.0 ± 0.2
Total corrections[d]	-0.57	-1.12	-0.40 ± 0.2	+0.09 ± 0.2
End-point energy (total), kev[e]	1293 ±1	2311 ±6.5	3713 ± 10	5011 ±30
$f(Z, E_m)$[f]	1.688	41.82	467.7	2036
$f_{\text{eff}}(Z, E_m)$, including all corrections	1.678	41.34	465.8	2038
Half-life, sec[e]	702 ±18	72.5 ±0.5	6.60 ± 0.06	1.53 ±0.02
$f_{\text{eff}}(Z, E_m)t$, sec	1178	2998 ±50	3074 ± 53	3118 ±110
G_V, 10^{-49} erg cm^3	1.35 ±0.04[g]	1.43 ±0.01	1.41 ± 0.01	1.40 ±0.01

[a]See reference 13.
[b]See reference 9.
[c]Taken from W. M. MacDonald, Phys. Rev. 110, 1420 (1958), for O^{14} and Cl^{34}, and estimated for Al^{26*} on the basis of those values. The calculation for Cl^{34} was incomplete, but the omitted terms are expected to decrease slightly the magnitude of the 1.2% correction quoted. We therefore assume a correction of (1.0 ± 0.2) %.
[d]The greatest theoretical uncertainties are probably in the Coulomb corrections to the nuclear matrix elements for Al^{26*} and Cl^{34} (see preceding remarks).
[e]The end-point energies and half-lives are those summarized by O. C. Kistner and B. M. Rustad, Phys. Rev. 114, 1329 (1959), with the exception of the O^{14} end point (see references 5 and 6).
[f]Calculated by numerical integration using Tables for the Analysis of Beta Spectra, National Bureau of Standards Applied Mathematics Series, No. 13 (U. S. Government Printing Office, 1952).
[g]Calculated using $(G_A/G_V)^2 = 1.56 \pm 0.14$, as determined from the value of the neutron spin-electron momentum correlation coefficient, $A = -0.11 \pm 0.02$ [M. T. Burgy, V. E. Krohn, T. B. Novey, G. R. Ringo, and V. L. Telegdi, Phys. Rev. Letters 1, 324 (1958)].

the infrared-divergent parts, as previously indicated.

In interpreting these results, one should note that the present approach necessitates the abandonment of the usual formulation of universality, based on the equality of bare coupling constants in a Lagrangian theory. In this sense, there is no compelling reason for believing in the equality of the renormalized constants G_V and G_μ as we have defined them. On the other hand, the necessity of avoiding infrared divergences evidently permits only one natural definition of the renormalized coupling constant, that which we have used. It is therefore of interest to inquire into the consequences of assuming that $G_V = G_\mu$. Using the value of G_V derived from the O^{14} data,[5,6] this assumption leads in fact to a value for the muon lifetime,

$$\tau_\mu = (2.23 \pm 0.05) \times 10^{-6} \text{ sec}, \tag{13}$$

in excellent agreement with experiment.[7] Further discussion of these results, and details of the techniques employed in the calculations, will be presented in a paper now being prepared.

We wish to acknowledge numerous valuable conversations with R. Blankenbecler, R. Behrends, A. Sirlin, and G. C. Wick. We also wish to thank I. Cole for computational aid.

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[1]R. P. Feynman and M. Gell-Mann, Phys. Rev. 109, 193 (1958); M. Gell-Mann, Phys. Rev. 111, 362 (1958).
[2]T. Kinoshita and A. Sirlin, Phys. Rev. 113, 1652 (1959); S. M. Berman, Phys. Rev. 112, 267 (1958); R. E. Behrends, R. J. Finkelstein, and A. Sirlin, Phys. Rev. 101, 866 (1956).
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[4]D. A. Bromley, E. Almqvist, H. E. Gove, A. E. Litherland, E. B. Paul, and A. J. Ferguson, Phys. Rev. 105, 957 (1957).

[5]R. O. Bondelid, J. W. Butler, C. A. Kennedy, and A. del Callar (to be published) (private communication from R. O. Bondelid).

[6]It is of considerable importance that the discrepancy between the $C^{12}(He^3, n)O^{14}$ threshold measurements of references 4 and 5 be resolved. In the present paper, we will use the $O^{14}(\beta^+)N^{14*}$ end-point kinetic energy of 1800 ± 6.5 kev calculated from the threshold value of reference 5.

[7]J. Fischer, B. Leontic, A. Lundby, R. Meunier, and J. P. Stroot, Phys. Rev. Letters 3, 349 (1959), $\tau_\mu = 2.20 \pm 0.015\ \mu$sec. R. A. Reiter, T. A. Romanski, and R. B. Sutton, post-deadline paper, Washington APS Meeting, 1960, $\tau_\mu = 2.211 \pm 0.003\ \mu$sec. V. L. Telegdi, R. A. Swanson, R. A. Lundby, and D. D. Yovanovitch (private communication from Dr. Yovanovitch), $\tau_\mu = 2.208 \pm 0.004\ \mu$sec.

[8]We are indebted to Professor R. H. Dalitz for pointing out to us that the results of the perturbation theory calculations for the muon are ambiguous to the extent that one is at liberty to regularize in different ways the muon and electron self-energy graphs and the vertex graphs, thereby adding finite constant terms to the decay amplitude. The choice of reference 2 corresponds to regularizing only the photon propagators.

[9]To lowest order in α, a consistent treatment of the decay accompanied by inner bremsstrahlung yields results identical with those of perturbation theory, with the usual weak-coupling constant G replaced by $\sqrt{2}\,a(s_0, \lambda)$.

[10]The factor $\sqrt{2}$ is introduced to conform to the notation for G of reference 1.

[11]M. Morita, Phys. Rev. 113, 1584 (1959).

[12]The failure to obtain a finite result for $A(s)$ when the electromagnetic form factor is included in the calculation of $\mathrm{Im}A(s)$ contradicts previous conjectures (see the first two papers of reference 2). We remark, however, that the form factor is necessary and sufficient to secure convergence in the integral for $B(s)$ and in the contributions to the neutron decay amplitude involving interactions of the electron with the anomalous magnetic moments of the nucleons. These form-factor-dependent terms do not contribute significantly to the corresponding decay rates.

[13]A term in the total electromagnetic correction of the form $-Z\alpha\pi/v_e$ is identified in the usual manner with the term of order $Z\alpha$ in the expansion of the Fermi factor for the positron decay, and is omitted from the corrections designated as radiative.

SOME APPLICATIONS OF THE GENERALIZED UNITARITY RELATION

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Recent studies of the analytic continuations of perturbation theory amplitudes have led to a greatly increased understanding of their properties.[1-10] Landau[4] showed the existence of a class of singularities which are a generalization of the ordinary thresholds for the onset of new physical processes. He derived two conditions for determining the location of these singularities: (1) Each of the intermediate particles (which correspond to lines joining otherwise disjunct subgraphs of the original Feynman graph) is on the mass shell. (2) The four-momenta of these intermediate particles satisfy a geometrical relation which generalizes the requirement, for the ordinary thresholds, that the intermediate particles all be at rest in the center-of-mass frame.

Applications of dispersion relations have all made use of the fact that the discontinuity across a branch cut starting from an ordinary physical threshold is obtained from the unitarity condition. It has been shown, for all finite order graphs, that a generalized unitarity relation gives the discontinuity across a branch cut which starts from any Landau singularity.[11] In this note we point out, by means of some simple examples, that this generalized unitarity relation greatly augments the power of analytic continuation methods.

The rule for calculating the discontinuity of a graph is that in the integral over the virtual four-momenta, for each line of the "reduced graph" which determines the singularity the factor $(M^2 - q^2)^{-1}$ is replaced by $2\pi i\delta(q^2 - M^2)$. In other words, the subgraphs which comprise the vertices of the reduced graph are considered only for the case that the lines leading into them represent free particles. The products of the factors $\delta(q^2 - M^2)$ and d^4k for the reduced graph are analogous to phase space volume elements.

Certain reduced graphs lead to poles rather than branch points; in these cases the residue is obtained by considering the lines to be on the mass shell. The poles discussed by Chew and Low[12] are simple examples.