HIGHER ORDER TERMS IN THE LAMB SHIFT CALCULATION

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Following previously described techniques,[1] the atomic energy level displacements of order $\alpha(Z\alpha)^6 \ln^2(Z\alpha)mc^2$ have been calculated and yield a result in agreement with that of the preceding Letter.[2] The procedure utilized represents a direct extension of the methods introduced in reference 1, wherein a counter-series, $\sum_{n=0}^{\infty} J_n$, having zero expectation value for the particular atomic state in question, is added to the series representing the iterative expansion of the bound-state electron propagator, $\sum_{n=0}^{\infty} I_n$. This addition corresponds to a change of gauge of the virtual photon defining the one-photon Lamb shift, and serves to remove spurious lower order contributions from each term of the combined sequence, $M_n = I_n + J_n$, thereby generating a simple calculational scheme for the lowest order Lamb shift and a particularly convenient method for the estimation of the magnitude of the Bethe logarithm.

It may also be of some interest to remark that the first relativistic corrections to the lowest order Lamb shift [of order $\alpha(Z\alpha)^5 mc^2$], and to hfs [of order $\alpha(Z\alpha)E_{\mathrm{H}}$], may both be obtained in a quite straightforward manner by the use of these techniques. In fact, the necessary matrix elements for the hfs calculation can be read off immediately from those of the similar Lamb shift effect. It is easily seen, for either case, that contributions of the desired order arise only from the terms $M_0 + M_1 + M_2$. A simple parametric decomposition uniquely isolates that portion of these matrix elements of the proper order (in $Z\alpha$); and the evaluation of elementary integrals then yields the correct, previously obtained[3] results.

Corrections of order $\alpha(Z\alpha)^6 \ln^2(Z\alpha)mc^2$ to the Lamb shift[4] arise from the matrix elements of $M_0 + M_1 + M_2 + M_3$, where the contributions of M_3 are all, to this order, gauge-variant, and serve to cancel similar gauge-variant terms remaining in the combination $M_0 + M_1 + M_2$. As a convenient check on the manipulations, all matrix elements may be grouped into gauge-invariant and gauge-variant combinations, and the latter explicitly shown to cancel. To this order in $Z\alpha$, it is necessary to include a relativistic correction to the ordinary S-state atomic wave functions $\phi_n(r)$; this essentially corresponds to averaging the latter, together with that portion of the operator which leads to the Bethe logarithm in the lowest order calculation, over distances comparable to the electron's Compton wavelength. To illustrate, consider the large component of the $2S$-state (Dirac) wave function which, correct to the order $\alpha(Z\alpha)^6 \ln^2(Z\alpha)$ in the final result, may be written in the form

$$\psi_2(r) = |\phi_2(0)| e^{-\gamma r}(1-\gamma r)(\gamma r)^{s-1},$$

where $s = [1-(Z\alpha)^2]^{1/2}$, and $\gamma = Z\alpha m/2$. Approximating the quantity $(\gamma r)^{s-1}$ by $1 - \frac{1}{2}(Z\alpha)^2 \ln(\gamma r)$, one obtains the desired correction

$$\Delta\phi_2(r) = -\tfrac{1}{2}(Z\alpha)^2 \ln(\gamma r)\phi_2(r).$$

In conjunction with that part of M_1 which yields the Bethe logarithm (corresponding to the infrared divergent part of the free-particle vertex operator), this wave function correction (wfc) will yield a relative factor of $(Z\alpha)^2 \ln(Z\alpha)$, since the corresponding matrix elements will contribute only when $r \sim m^{-1}$ (momentum $\sim m$, rather than $Z\alpha m$).

However, the major contributions of this order arise from the structure of the operators themselves; in essence one has to deal with integrals of the form[5]

$$I_{n,l} = \int dT \int_0^1 dx\, x^e [D_n(\lambda)]^{-(l+1)}, \qquad (1)$$

where the index n specifies the parent matrix element, viz.:

$$D_1(\lambda) = xm^2 + (1-x)\Delta_y + x\lambda y(1-y)q^2$$

appears in the terms of M_1. The parameter λ may have any positive value, including zero; and $\int dT$ stands for some combination of the di-

mensionless momenta, obtained from those matrix elements of the correct nominal order (without wfc), which would be logarithmically divergent without the convergence factors $D_n(\lambda)$. Omitting that part of M_1 which leads to the wfc contribution, the gauge-invariant terms which appear in M_0+M_1 may be written as

$$\tfrac{2}{3}(\vec{p}_1\times\vec{p})^2\vec{a}+2im\vec{a}\vec{a}(\tfrac{5}{12}\vec{P}^2-\tfrac{1}{4}\vec{Q}^2),$$

while those of M_2 are

$$2im\vec{a}\vec{a}(-\tfrac{1}{12}\vec{P}^2-\tfrac{1}{4}\vec{Q}^2),$$

where P, Q denote the "gauge-invariant" momentum combinations

$$\vec{P}^2=2\vec{p}'^2-\vec{p}_1^{\,2}-\vec{p}^2,\quad \vec{Q}^2=(\vec{p}_1-\vec{p}')^2+(\vec{p}-\vec{p}')^2-(\vec{p}_1-\vec{p})^2.$$

The integrations over momenta and the parametric variables, as indicated in Eq. (1), have been suppressed; it is found that the $\ln^2(Z\alpha)$ contributions of such integrals are independent of l,n, and the denominator "y variables." These contributions may conveniently be determined by noting that if one anticipates $Z\alpha$ dependence of the form

$$I(Z\alpha)=A\ln^2(Z\alpha)+B\ln(Z\alpha)+C+D(Z\alpha)+\ldots,$$

the desired coefficient, A, may be obtained by calculating

$$\tfrac{1}{2}\left[(Z\alpha)\frac{\partial}{\partial(Z\alpha)}\left((Z\alpha)\frac{\partial I(Z\alpha)}{\partial(Z\alpha)}\right)\right]_{Z\alpha\to 0}.$$

The resulting integrals are vastly simpler to evaluate than are the original $I(Z\alpha)$; and in this way it is not difficult to obtain the final answer (nonzero for S states only):

$$\Delta E=\frac{4}{3\pi n^3}\alpha(Z\alpha)^6\ln^2(Z\alpha)mc^2\{1-4\}. \qquad (2)$$

The first term in the bracket of Eq. (2) represents the wfc contribution; the second one arises from elements of the self-energy operator, as outlined above. The numerical effect of this result on the theoretical estimate of the Lamb shift for H, D, and He^+ is presented in the preceding Letter.

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[1]H. M. Fried and D. R. Yennie, Phys. Rev. 112, 1391 (1958).

[2]A. Layzer, preceding Letter [Phys. Rev. Letters 4, 580 (1960)].

[3]R. Karplus, A. Klein, and J. Schwinger, Phys. Rev. 86, 288 (1952); M. Baranger, H. A. Bethe, and R. P. Feynman, Phys. Rev. 92, 482 (1953); N. M. Kroll and F. Pollock, Phys. Rev. 86, 876 (1952); R. Karplus, A. Klein, and J. Schwinger, Phys. Rev. 84, 597 (1951).

[4]The probable existence of such corrections was pointed out several years ago by E. Wichmann (private communication).

[5]We use the notation of reference 1.