sents the failure of the resonance approximation for large $|s|$; in the resonance region, the effect of the pole is negligible if $\nu_0$ is sufficiently large and $\Gamma$ is not too large. The distinction between the two viewpoints may be seen by noting that if the pole is interpreted as a phenomenological representation of the left-hand branch cut, then $\nu_0 \sim m^2$, where $m$ is some average mass of the intermediate states which contribute to the left-hand branch cut. Since $\nu_0 \sim 100 - 1000$ for values of the width $\Gamma$ which fit the data on nucleon electromagnetic structure,[2] one is led to suspect that virtual baryon-antibaryon pairs play a prominent role in producing the $p$-wave resonance,[10] unless an unstable vector boson such as Sakurai and others have suggested[6] is present in the original field-theoretical Lagrangian.

If Eq. (10), with the pole removed, is inserted into the iterative procedure of Chew et al.,[4] then the resonance should persist, and, in addition, nonresonant terms will appear in other angular momentum states; in particular, one should be able to calculate the $s$-wave scattering lengths in terms of the resonance parameters. It also seems clear that a purely dispersion theoretic approach will require at least two parameters to obtain a $p$-wave resonance, since the one-parameter theory of Chew et al., which corresponds to a $\lambda\phi^4$ pion-pion interaction,[11] seems incapable of producing a resonance. Whether these two parameters are to be related to the pion-pion and pion-nucleon coupling constants, or to the mass and coupling constant of the conjectured vector boson, cannot be decided on the basis of present knowledge.

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†Harrison Special Fellow.

[1]W. Frazer and J. Fulco, Phys. Rev. Letters 2, 365 (1959).

[2]W. Frazer and J. Fulco, Phys. Rev. 117, 1603, 1609 (1960).

[3]G. Chew and S. Mandelstam, Lawrence Radiation Laboratory Report UCRL-8728; Phys. Rev. (to be published).

[4]G. Chew, S. Mandelstam, and H. P. Noyes, Lawrence Radiation Laboratory Report UCRL-9001; Phys. Rev. (to be published).

[5]S. Mandelstam, Phys. Rev. 112, 1344 (1958).

[6]As has recently been suggested by J. Sakurai, Ann. Phys. (to be published). See also C. Yang and R. Mills, Phys. Rev. 96, 191 (1954).

[7]L. Castillejo, R. Dalitz, and F. Dyson, Phys. Rev. 101, 453 (1956).

[8]This point is discussed for the static theory by R. Norton and A. Klein, Phys. Rev. 109, 991 (1958).

[9]More precisely, $\Gamma' = \pi\Gamma(\pi + \xi\Gamma)^{-1}$, $\Gamma'\nu_\gamma' = \Gamma'\nu_\gamma K(\nu_\gamma)$, where $\nu_\gamma'$, $\Gamma'$ are the quantities designated as $\nu_\gamma$, $\Gamma_\gamma$ by Frazer and Fulco.[1,2]

[10]K. Igi and K. Kawarabayashi, Progr. Theoret. Phys. (Kyoto) 20, 576 (1958), have noted that the contribution to the $p$-wave pion-pion scattering amplitude from a single nucleon closed loop corresponds to an attractive potential.

[11]S. Okubo, Phys. Rev. 118, 357 (1960).

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## NEW THEORETICAL VALUE FOR THE LAMB SHIFT*

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Although the good agreement between theory and experiment for the Lamb shift is one of the notable successes of quantum electrodynamics, the most recent tabulations[1] still have shown small but significant discrepancies for H, D, and $He^+$ which have made a further increase in the accuracy of the theoretical value desirable. Since the listed discrepancy for $He^+$ is roughly 60 times that of H or D, it was natural to try to reduce this disagreement by calculating the second order radiative correction of order $\alpha(\alpha Z)^6$, which is the next order in $(\alpha Z)$, the Coulomb interaction parameter, beyond the previously calculated[2,3] order of $\alpha(\alpha Z)^5$.

A closer mathematical analysis[4,5] shows that there exist the leading orders of $\alpha(\alpha Z)^6\ln^2(\alpha Z)$ and $\alpha(\alpha Z)^6\ln(\alpha Z)$. We have completed the calculation of these two orders. The analytic result is

$$\Delta E(2S - 2P_{1/2}) = -Lw[\tfrac{3}{4}\ln^2 w + \ln w(4\ln 2 + 1 + 7/48)], \quad (1)$$

where $\Delta E(2S - 2P_{1/2})$ is the difference of the shifts of the $2S$ and $2P_{1/2}$ levels due to the two new orders, $w \equiv (\alpha Z)^2$, and $L$ is $Z^4$ times the "Lamb constant": $L \equiv Z^4\alpha^3/(3\pi)$ ry $= Z^4(135.6)$ Mc/sec. The $\alpha w^3\ln^2 w$

coefficient vanishes for non-$S$ states and is proportional to $1/n^3$ for the $nS$ state. The value of the $\alpha w^3 \ln^2 w$ coefficient has been recently confirmed by an independent calculation of Fried and Yennie.[6]

The corresponding numerical results in Mc/sec are -0.25 for H or D and -9.5 for $He^+$. These additions reduce markedly the disagreement between theory and experiment and, when added to the theoretical values listed by Petermann,[1] lead to the new theoretical values listed in Table I, which are compared with experimental values for H, D, and He[4]. Theoretical errors are copied from Petermann's article.

The theoretical foundation of the calculation of the new orders has been briefly described in reference 4. The method used is the free-propagator expansion, the algebraic expansion of the bound electron propagator or Green's function in "powers" of the Coulomb potential. This expansion was first employed in the Lamb shift problem by Fried and Yennie[5] who successfully extracted the two lowest orders of $L\ln w$ and $L$ with the aid of a special photon gauge. In our method, this gauge is not employed and the photon propagator is of the usual Feynman form proportional to $1/k^2$.

Details of the calculation of the two new orders will be published later along with more general results of a mathematical nature concerning some properties of the free-propagator expansion connected with an expansion of the self-energy in orders of $(\alpha Z)$. The remainder of this report is devoted to a further presentation of the results of the calculation for arbitrary bound states and in particular for the $1S$, $2S$, $2P_{1/2}$, and $2P_{3/2}$ states.

Let us introduce the notation

$$\psi^2(0) \equiv |\int d^3p \langle p|u\rangle|^2 = 8\pi^3 |\langle x=0|u\rangle|^2, \quad (2)$$

$$\psi^2(0)/(\pi^2) = 8/(n^3) \text{ for } nS \text{ state}$$

$$= 0 \text{ for } L \neq 0, \quad (3)$$

where $u$ is the nonrelativistic Schrödinger wave function in "$Z$-atomic" units ($\hbar = c = \alpha Z m = 1$). These units will be understood in the following.

The shift $\Delta E_1$ due to both the $\alpha w^3 \ln^2 w$ and $\alpha w^3 \ln w$ terms can be conveniently expressed as

$$\Delta E = \Delta E_1 + \Delta E_2, \quad (4)$$

where $\Delta E_1$ is of order $\alpha w^3 \ln w$ and proportional to $\psi^2(0)$, and $\Delta E_2$ is the sum of a number of matrix elements which can be easily evaluated to the desired order in $w$ for the particular bound state under consideration. $\Delta E_2$ contains the entire $\alpha w^3 \ln^2 w$ coefficient, proportional to $\psi^2(0)$, and the remainder of the $\alpha w^3 \ln w$ coefficient, which is not proportional to $\psi^2(0)$.

$\Delta E_1$ is given by

$$\Delta E_1 = -L\frac{\psi^2(0)}{\pi^2} w \ln w (4\ln 2 - \tfrac{47}{16} - \tfrac{1}{10} = -0.26). \quad (5)$$

The last term in the parenthesis, $-1/10$, is due to vacuum polarization and can be easily derived as the effect of the Dirac modification of the large-component wave function on the expectation value of the Uehling potential.

The remainder of $\Delta E$ is given by ($V = -1/r$)

$$\Delta E_2 = -Lw\int^w \frac{dw}{w} \times [-4\langle u'|\ldots q^2 V|u\rangle - \tfrac{1}{2}i\langle \sigma_{ij} p_i q^2 V p_j\rangle - (11/10)\langle p_i \ldots q^2 V \ldots p_i\rangle + (8/5)\langle p^2 \ldots q^2 V\rangle + 2\langle p^2 \ldots [p_i, V] p_i\rangle - 6\langle [p_i, V] \ldots [p_i, V]\rangle]. \quad (6)$$

Here $u'$ is the first "Dirac correction" to the expansion in powers of $w$ of the exact large-component wave function $\psi$: $\psi \simeq u + wu'$. The angular brackets $\langle\ \rangle$ denote the expectation value with respect to the nonrelativistic Schrödinger wave functions $u$. The three dots denote a convergence factor of $(1+wp^2)^{-1}$. The lower limit of integration is arbitrary but finite and fixed ($w$-independent).[7]

The evaluation of $\Delta E_2$ for the $1S$, $2S$, $2P_{1/2}$, and $2P_{3/2}$ states yields, with the omission of the

Table I. Theoretical and experimental values of the Lamb shift in Mc/sec for H, D, and $He^+$.

| | H | D[a] | $He^+$ [b] |
|---|---|---|---|
| Theoretical | 1057.70 ± 0.15 | 1058.96 ± 0.16 | 14046.3 ± 3.0 |
| Experimental | 1057.77 ± 0.10 | 1059.00 ± 0.10 | 14040.2 ± 4.5 |

[a] S. Triebwasser, E. Dayhoff, and W. Lamb, Phys. Rev. 89, 98 (1953).
[b] E. Lipworth and R. Novick, Phys. Rev. 108, 1434 (1957).

previously given $\alpha w^3 \ln^2 w$ term,

$$\Delta E_2(1S) = -Lw\ln w(24\ln 2 + 86/5 = 33.8), \quad (7)$$

$$\Delta E_2(2S) = -Lw\ln w(369/80 = 4.61), \quad (8)$$

$$\Delta E_2(2P_{1/2}) = -Lw\ln w(103/240 = 0.43), \quad (9)$$

$$\Delta E_2(2P_{1/2} - 2P_{3/2}) = -Lw\ln w(-3/16 = -0.19). \quad (10)$$

The contribution to the fine structure separation $2P_{1/2} - 2P_{3/2}$ is due, of course, to the spin-orbit matrix element of (6).

We conclude this report with a brief speculation about the magnitude of the uncomputed term of order $wL$, which is technically of higher order than the orders calculated here through the absence of factors of $\ln w$. Since the coefficient of the $\ln w$ term in the $2S - 2P_{1/2}$ shift given in (1) is about 5 times as large as the $\ln^2 w$ coefficient and ($-\ln w$) is only about 10 ($8\frac{1}{2}$) for H ($He^+$), the contribution of the $\ln w$ term is roughly 1/2 (3/5) that of the $\ln^2 w$ term for H ($He^+$) with opposite sign. A comparable ratio of the "constant" to the $\ln w$ contribution would mean that the term of order $wL$ is still important.

There are theoretical reasons, however, for believing that this is too dark a picture and that the "constant" term is only about 10% (20%) of the shift calculated here for H ($He^+$) due to the logarithmic terms. Let us note first that the $\ln^2$ and ln contributions will change if one picks $w' = \lambda w$ rather than $w$ as the argument of the logarithmic factors, where $\lambda$ is an arbitrary $w$-independent number, a substitution which corresponds to the choice of $w'$ rather than $w$ as the "natural" expansion parameter.[8] Thus, $A\ln^2 w + B\ln w + C = A\ln^2 w' + B(\lambda)\ln w' + C(\lambda)$, where $B(\lambda) = B - 2A\ln\lambda$ and $C(\lambda) = C + A\ln^2\lambda - B\ln\lambda$.

If one assumes that the uncalculated "constant" term, $C(\lambda)$, is negative for $\lambda = 1$, which is in accordance with the fact that all five previous orders in $w$ alternate in sign for $\lambda = 1$ (and $w < 1$), then it is rigorously true that the "constant" term is of minimum absolute value (and $\leq 0$) for some value of $\lambda$ greater than unity but less than $\lambda^*$ where $\lambda^*$ is such that the coefficient, $B(\lambda)$, of the $\ln w'$ term vanishes. In our case, $\lambda^* = e^{2.6} = 13$. With the choice $\lambda = \lambda^*$ one finds that the contribution in Mc/sec to the Lamb shift due to the logarithmic terms is changed from -0.25 to -0.28 and from -9.5 to -11.7 for H and $He^+$, respectively.[9] It is satisfying that the change is small and in a direction towards improving the over-all agreement with experiment.

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[1]A. Petermann, Fortschr. Physik 6, 505 (1958). We refer to this article also for a discussion and classification of the numerous theoretical effects.

[2]M. Baranger, H. A. Bethe, and R. P. Feynman, Phys. Rev. 92, 482 (1953).

[3]R. Karplus, A. Klein, and J. Schwinger, Phys. Rev. 86, 876 (1952).

[4]A. Layzer, Bull. Am. Phys. Soc. 4, 280 (1959).

[5]H. M. Fried and D. R. Yennie, Phys. Rev. 112, 1391 (1958).

[6]H. M. Fried and D. R. Yennie, following Letter [Phys. Rev. Letters 4, 583 (1960)].

[7]The spin-orbit and $\langle p_i \ldots q^2 V \ldots p_i \rangle$ matrix elements vanish for $S$ states. The Dirac and $\langle p^2 \ldots q^2 V \rangle$ matrix elements vanish for non-$S$ states. For non-$S$ states the convergence factor may be eliminated and $\int^w dw/w$ replaced by $\ln w$. For $S$ states, the nonvanishing Schrödinger matrix elements diverge like $w^{-1/2}$ in the limit $w \to 0$ and therefore give rise to a lower order $\alpha w^{5/2}$ contribution to $\Delta E$ that should be discarded. [The $w^{-1/2}$ part of the matrix elements could be eliminated by replacing $w$ by $-w + i\epsilon$ in the convergence factors $(1 + wp^2)^{-1}$ and taking the real part of the matrix elements.] The $\alpha w^3 \ln^2 w$ contribution is due to matrix elements with a $\ln w$ divergence for $S$ states. This divergence is proportional to $\psi^2(0)$. The corresponding matrix elements are the Dirac matrix element and the two Schrödinger matrix elements involving $[p_i, V]$.

[8]A similar situation arises for the lowest order contributions of order $\alpha w^2 \ln w$ ($\sim L\ln w$) and $\alpha w^2$ ($\sim L$). There, as is well-known, it is natural to introduce an "average excitation energy" for the $2S$ state of magnitude $(17.7)/2 = 8.8$ and the term of order $\alpha w'^2$ can be made to vanish for a choice of $\lambda$ nearly equal to this excitation energy.

[9]Virtually the same results would have been obtained for the choice $\lambda = 8.8$ corresponding to the $2S$ average excitation energy.