from this solution by specializing p as before, substituting $\rho=\lambda^{-2}+\lambda^{-1}\tilde{\rho}$, $\sigma=\lambda\tilde{\sigma}$, $\xi=\lambda^2\tilde{\xi}$, $\eta=\lambda^2\tilde{\eta}$, $q=\lambda^4\tilde{q}$, where λ is constant, and taking the limit as λ tends to zero. There is then a singularity on every wave front where the homogeneity conditions $\partial^3H/\partial\xi^3=\partial^3H/\partial\eta^3=0$ are violated.

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POISSON BRACKETS BETWEEN LOCALLY DEFINED OBSERVABLES IN GENERAL RELATIVITY*

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In this note we describe a new method for formulating local observables, and Poisson brackets between them, in Einstein's theory of gravitation. The applicability of this method depends on the functional independence of the four scalars of the Weyl tensor but involves no global or topological assumptions. It combines the Hamiltonian approaches of Dirac[1] with the construction of observables by Komar,[2] and it leads to closed-form expressions both for the observables and the Poisson brackets.

Our point of departure is the discovery that the four scalars of the Weyl tensor can be expressed in closed form in terms of the 12 canonical variables g_{mn}, p^{mn} as defined by Dirac. Using the notation,

$$v_{mn}=(g^{00})^{-1/2}\left\{ {0 \atop m\ n} \right\}, \tag{1}$$

to denote Dirac's "invariant velocities," we find for the components of the Weyl tensor the following expressions:

$$C_{iklm}={}^3R_{iklm}+v_{kl}v_{im}-v_{il}v_{km},$$
$$C_{ikl}\equiv C_{ikl\rho}l^\rho=v_{li/k}-v_{lk/i},$$
$$C_{kl}\equiv C_{\rho kl\sigma}l^\rho l^\sigma=-e^{im}C_{iklm}. \tag{2}$$

${}^3R_{iklm}$ denotes the three-dimensional curvature tensor, and the solidus signifies covariant differentiation with the help of three-dimensional Christoffel symbols. The Weyl scalars are quadratic and cubic expressions in these components. For instance, the first two (the quadratic) scalars are

$$A^1=C_{iklm}C^{iklm}+4C_{ikl}C^{ikl}+4C_{kl}C^{kl},$$
$$A^2=\epsilon^{lms}(C_{iklm}C^{ik}{}_s+2C_{klm}C^k{}_s). \tag{3}$$

ϵ^{lms} is Levi-Civita's (three-dimensional) fully antisymmetric tensor, and indices are raised and lowered with the help of the three-dimensional metric e^{mn}, g_{mn}.

In order to retain flexibility, we shall assume that the four intrinsic coordinates to be used will be some four functions $f^\rho(A^1,\ldots,A^4)$ whose specification we may reserve; if desired they may be chosen so that the f coordinates are asymptotically Lorentzian for some specific Riemann-Einstein manifold. We shall, accordingly, introduce the four coordinate conditions

$$f^\rho-x^\rho=0, \tag{4}$$

which determine the coordinate system uniquely. These four coordinate conditions, along with the constraints

$$H_s=0,\ H_L=0, \tag{5}$$

form a system of eight second-class constraints, and hence lend themselves to the construction of field variables and functionals whose Poisson brackets with all the constraints (4), (5) vanish. Such variables are then observables in the technical sense in which this expression is used now in general relativity. They are identical with the observables constructed in reference 2, but now they appear as closed-form expressions in terms of Dirac's 12 canonical field variables.

As a preliminary we note that the remaining components of the metric tensor are uniquely determined by the coordinate conditions, and that they are closed-form expressions in terms of the canonical variables. We have

$$(g^{00})^{1/2}\delta(x,x') = [f^0, H_L'],$$

$$g^{0n}\delta(x,x') = (g^{00})^{1/2}[f^n, H_L']. \qquad (6)$$

For the consistency of the whole scheme it is of the essence that the Poisson brackets on the right contain only the delta function as a factor, and not, e.g., derivatives of the delta function. This condition is satisfied automatically, because the constraints (5) are generators of infinitesimal coordinate transformations, and the f^ρ, being scalars, have transformation laws that consist solely of the transport term,

$$\bar{\delta} f^\rho = -f^\rho{}_{,\sigma}\delta x^\sigma. \qquad (7)$$

Since the right-hand side contains the δx^σ only undifferentiated, no derivatives of the delta function will occur on the right of Eqs. (6).

To construct an observable we start from any functional of the canonical variables, A. We shall now add to A a linear combination of the constraints (4), (5), so that the new functional A^* commutes with all of them,

$$A^* = A + \int [\gamma_\rho(f^\rho - x^\rho) + \epsilon^s H_s + \epsilon^L H_L] d^3x. \qquad (8)$$

Provided the constraints are satisfied, A^* will equal in value the original A. The coefficients γ, ϵ are determined as follows:

$$\gamma_k = [H_k, A],$$

$$g^{00}\gamma_0 = (g^{00})^{1/2}[H_L, A] - g^{0s}\gamma_s,$$

$$\epsilon^k = [f^k, A] + \int [f^k, f^{\rho\prime}]\gamma_\rho{}' d^3x',$$

$$(g^{00})^{1/2}\epsilon^L = [f^0, A] + \int [f^0, f^{\rho\prime}]\gamma_\rho{}' d^3x'. \qquad (9)$$

The inversion of the matrix of the Poisson brackets of the second-class constraints is thus accomplished locally and without difficulty. The Poisson brackets between any two observables A^*, B^* can now be calculated by means of the conventional Poisson brackets between canonical field variables.

The Poisson bracket between A^* and the Hamiltonian vanishes, because the Hamiltonian is but a linear combination of the constraints (5), as was originally shown by Dirac.[1] Nevertheless, A^* is not a constant of the motion, but rather we have

$$\frac{dA^*}{dt} = \frac{\partial A^*}{\partial t} = -\int \gamma_0 d^3x. \qquad (10)$$

A fuller account of our results will be submitted to the Physical Review.

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