SPHERICAL GRAVITATIONAL WAVES*

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This note presents a class of solutions to Einstein's gravitational equations for empty space. Some of the solutions appear to represent a very simple kind of spherical radiation. The metric considered has the form

$$ds^2 = 2d\rho\, d\sigma + (K - 2H\rho - 2m/\rho)d\sigma^2$$
$$- \rho^2 p^{-2}\{[d\xi + (\partial q/\partial\eta)d\sigma]^2 + [d\eta + (\partial q/\partial\xi)d\sigma]^2\},$$

where m is a function of σ only, p and q are functions of σ, ξ, η,

$$H = p^{-1}\partial p/\partial\sigma + p\partial^2 p^{-1}q/\partial\xi\partial\eta - pq\partial^2 p^{-1}/\partial\xi\partial\eta,$$

and K is the Gaussian curvature of the surface $\rho = 1$, $\sigma = $ constant,

$$K = p^2(\partial^2/\partial\xi^2 + \partial^2/\partial\eta^2)\ln p.$$

For this metric, the empty-space condition $R_{ik} = 0$ reduces to

$$\partial^2 q/\partial\xi^2 + \partial^2 q/\partial\eta^2 = 0$$

(from which we can derive $q = 0$ by a coordinate transformation), and

$$\partial^2 K/\partial\xi^2 + \partial^2 K/\partial\eta^2 = 4p^{-2}(\partial/\partial\sigma - 3H)m.$$

If these equations are satisfied, the curvature tensor may be written as

$$R_{ijkl} = \rho^{-3}D_{ijkl} + \rho^{-2}\text{III}_{ijkl} + \rho^{-1}N_{ijkl},$$

where D_{ijkl}, III_{ijkl}, and N_{ijkl} are tensors of type I degenerate, type III, and type II null, respectively. They are covariantly constant on any ray of constant σ, ξ, η.

The solution is degenerate type I if m is nonzero and K is independent of ξ, η. It is then reducible to $m = 1$, $p = \cosh\mu\xi$, $q = 0$, where μ is a real or purely imaginary constant. If μ is real and nonzero, this is Schwarzschild's solution for a mass μ^{-3}.

If $(\partial K/\partial\xi)^2 + (\partial K/\partial\eta)^2 \neq 0$ and the empty-space equations are satisfied, the metric is type II non-null or type III, the condition for type III being $m = 0$. As an example of these types, we mention $m = $ constant, $p = \xi^{3/2}$, $q = 0$. This contains a re-

gion of incorrect signature, which can be eliminated, however, by the substitution $\xi = 4\zeta^{-2}$.

In the remaining case, where m vanishes and K is independent of ξ, η, the solution is type II null or flat, the condition for flatness being $A = 0$, where

$$A = (\partial/\partial\xi + i\partial/\partial\eta)[p^2(\partial/\partial\xi + i\partial/\partial\eta)H].$$

For K nonzero, $|A/K\rho|$ is invariant under all coordinate transformations which preserve the form of the metric. It might be described as the intensity of the gravitational field.

This metric and the plane gravitational wave have the same local geometry, since their curvature tensors are algebraically indistinguishable. The wave fronts are subspaces of constant σ. In the linear approximation, they are spheres expanding at the speed of light, the source having a velocity less than, equal to, or greater than this, according as K is positive, zero, or negative.

The coordinates ξ and η may be chosen so that $p = 1 + \frac{1}{4}(\xi^2 + \eta^2)K(\sigma)$. All components of the curvature tensor—covariant, contravariant, and mixed—are then homogeneously linear in q. The contravariant metric tensor may be divided into a flat background metric, independent of q, and a residue which is homogeneously linear in q. There is thus an uncommonly close relation between the rigorous solution and its linear approximation.

It is easy to construct nontrivial solutions which are periodic in σ. In such a solution, there is no monotonic change which could be identified as loss of energy by the source. For positive K, however, there is at least one singularity on any wave front where the field does not vanish identically; and P. G. Bergmann has suggested that this might conceivably represent a flow of matter which restores to the source the energy carried away by radiation.

We can derive the plane-fronted wave,[1]

$$ds^2 = 2d\rho\, d\sigma - 2Hd\sigma^2 - d\xi^2 - d\eta^2,$$
$$\partial H/\partial\rho = \partial^2 H/\partial\xi^2 + \partial^2 H/\partial\eta^2 = 0,$$

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from this solution by specializing p as before, substituting $\rho = \lambda^{-2} + \lambda^{-1}\tilde{p}$, $\sigma = \lambda\tilde{\sigma}$, $\xi = \lambda^2\tilde{\xi}$, $\eta = \lambda^2\tilde{\eta}$, $q = \lambda^4\tilde{q}$, where λ is constant, and taking the limit as λ tends to zero. There is then a singularity on every wave front where the homogeneity conditions $\partial^3 H/\partial\xi^3 = \partial^3 H/\partial\eta^3 = 0$ are violated.

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[1]H. W. Brinkmann, Math. Annalen 94, 119 (1925).

POISSON BRACKETS BETWEEN LOCALLY DEFINED OBSERVABLES IN GENERAL RELATIVITY*

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In this note we describe a new method for formulating local observables, and Poisson brackets between them, in Einstein's theory of gravitation. The applicability of this method depends on the functional independence of the four scalars of the Weyl tensor but involves no global or topological assumptions. It combines the Hamiltonian approaches of Dirac[1] with the construction of observables by Komar,[2] and it leads to closed-form expressions both for the observables and the Poisson brackets.

Our point of departure is the discovery that the four scalars of the Weyl tensor can be expressed in closed form in terms of the 12 canonical variables g_{mn}, p^{mn} as defined by Dirac. Using the notation,

$$v_{mn} = (g^{00})^{-1/2}\begin{Bmatrix} 0 \\ m \ n \end{Bmatrix}, \tag{1}$$

to denote Dirac's "invariant velocities," we find for the components of the Weyl tensor the following expressions:

$$C_{iklm} = {}^3R_{iklm} + v_{kl}v_{im} - v_{il}v_{km},$$

$$C_{ikl} \equiv C_{iklp}l^\rho = v_{li/k} - v_{lk/i},$$

$$C_{kl} \equiv C_{\rho kl\sigma}l^\rho l^\sigma = -e^{im}C_{iklm}. \tag{2}$$

${}^3R_{iklm}$ denotes the three-dimensional curvature tensor, and the solidus signifies covariant differentiation with the help of three-dimensional Christoffel symbols. The Weyl scalars are quadratic and cubic expressions in these components. For instance, the first two (the quadratic) scalars are

$$A^1 = C_{iklm}C^{iklm} + 4C_{ikl}C^{ikl} + 4C_{kl}C^{kl},$$

$$A^2 = \epsilon^{lms}(C_{iklm}C^{ik}{}_s + 2C_{klm}C^k{}_s). \tag{3}$$

ϵ^{lms} is Levi-Civita's (three-dimensional) fully antisymmetric tensor, and indices are raised and lowered with the help of the three-dimensional metric e^{mn}, g_{mn}.

In order to retain flexibility, we shall assume that the four intrinsic coordinates to be used will be some four functions $f^\rho(A^1, \ldots, A^4)$ whose specification we may reserve; if desired they may be chosen so that the f coordinates are asymptotically Lorentzian for some specific Riemann-Einstein manifold. We shall, accordingly, introduce the four coordinate conditions

$$f^\rho - x^\rho = 0, \tag{4}$$

which determine the coordinate system uniquely. These four coordinate conditions, along with the constraints

$$H_s = 0, \quad H_L = 0, \tag{5}$$

form a system of eight second-class constraints, and hence lend themselves to the construction of field variables and functionals whose Poisson brackets with all the constraints (4), (5) vanish. Such variables are then observables in the technical sense in which this expression is used now in general relativity. They are identical with the observables constructed in reference 2, but now