1841 (1972) [Sov. Phys. JETP 34, 979 (1972)].
[9]A. A. Ivanov and L. I. Rudakov, Zh. Eksp. Teor. Fiz. 58, 1332 (1970 [Sov. Phys. JETP 31, 715 (1970)].
[10]E. P. Lee and L. D. Pearlstein, Phys. Fluids 16, 904 (1973).
[11]H. Alfvên, Phys. Rev. 55, 425 (1939).

Critical Exponents for the n-Vector Model in Three Dimensions from Field Theory

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We present a new calculation of the critical exponents of the n-vector model through field-theoretical methods. The coefficients of the renormalization functions of the $(\vec{\varphi}^2)^2$ theory are expanded in powers of the coupling constant. Asymptotic estimates of large order of perturbation series are used to transform the divergent perturbation series into a convergent one. As a consequence, new and more precise values of critical exponents are obtained.

Initiated by Wilson,[1] the field-theoretical approach to critical phenomena[2] has been extremely successful in the domain of phase transitions. In particular, it has been possible to calculate physical quantities such as critical exponents through the famous Wilson-Fisher[3] $\epsilon = 4-d$ expansion. More recently, use has been made of perturbation series for the $g\varphi^4$ field theory directly in three dimensions[4,5] to calculate critical exponents for Ising-like systems.

We want to show here how recent progress in field theory—i.e., the asymptotic estimate of the behavior of perturbation series at large orders[6,7] —allows us to calculate more accurately the same exponents. The perturbative expansion for the renormalization-group functions of the $g(\vec{\varphi}^2)^2$ field theory with O(n) symmetry has now also been generated for $n=0$, 2, and 3 (Table I),[8] and we present, therefore, results for the n-vector model. The results for $n=0$ describe the statistics of polymers.[9]

The main ingredient that we shall use in the analysis of the perturbation series is the following: It can be shown[6,7] that in the $g(\vec{\varphi}^2)^2$ field theory large orders of the perturbative expansion of any physical quantity $A(g)$, with

$$A(g) = \sum_K A_K g^K, \tag{1}$$

ehave for large K $(K \to +\infty)$ as

$$A_K \sim K!(-a)^K K^b, \tag{2}$$

re a and b have been calculated in three dinsions[7] for the various renormalization-group

TABLE I. Perturbative expansion (Ref. 8) of renormalization-group functions of the $g(\vec{\varphi}^2)^2$ field theory with O(n) symmetry in three dimensions.

$n = 1$

$$W(g) = -g + g^2 - 0.4224965707\,g^3 + 0.3510695978\,g^4 - 0.376526828\,g^5 + 0.49554751\,g^6 - 0.749689\,g^7$$

$$\gamma^{-1}(g) = 1 - \frac{1}{6}g + \frac{1}{27}g^2 - 0.0230696213\,g^3 + 0.0198868203\,g^4 - 0.02245952\,g^5 + 0.0303679\,g^6$$

$$\eta(g) = 0.0109739369\,g^2 + 0.0009142223\,g^3 + 0.0017962229\,g^4 - 0.00065370\,g^5 + 0.0013878\,g^6$$

$n = 2$

$$W(g) = -g + g^2 - 0.4029629630\,g^3 + 0.3149169420\,g^4 - 0.317928484\,g^5 + 0.39110247\,g^6 - 0.552448\,g^7$$

$$\gamma^{-1}(g) = 1 - \frac{1}{5}g + \frac{1}{25}g^2 - 0.0259419075\,g^3 + 0.0205323538\,g^4 - 0.02219865\,g^5 + 0.0279829\,g^6$$

$$\eta(g) = 0.0118518519\,g^2 + 0.0009873601\,g^3 + 0.0018368107\,g^4 - 0.00058633\,g^5 + 0.0012514\,g^6$$

$n = 3$

$$W(g) = -g + g^2 - 0.3832262015\,g^3 + 0.2829466813\,g^4 - 0.270333298\,g^5 + 0.31255586\,g^6 - 0.414861\,g^7$$

$$\gamma^{-1}(g) = 1 - \frac{5}{22}g + \frac{5}{121}g^2 - 0.0276673019\,g^3 + 0.0201190591\,g^4 - 0.02101293\,g^5 + 0.0247497\,g^6$$

$$\eta(g) = 0.0122436486\,g^2 + 0.0010200001\,g^3 + 0.0017919258\,g^4 - 0.00050410\,g^5 + 0.0010883\,g^6$$

$n = 0$

$$W(g) = -g + g^2 - 0.4398148149\,g^3 + 0.3899226895\,g^4 - 0.447316097\,g^5 + 0.63385550\,g^6 - 1.034928\,g^7$$

$$\gamma^{-1}(g) = 1 - \frac{1}{8}g + \frac{1}{32}g^2 - 0.0184623402\,g^3 + 0.0172838882\,g^4 - 0.02062072\,g^5 + 0.0299914\,g^6$$

$$\eta(g) = 0.0092592593\,g^2 + 0.0007713750\,g^3 + 0.0015898706\,g^4 - 0.00066062\,g^5 + 0.0014103\,g^6$$

functions [$W(g)$, $\eta(g)$, etc.]. The presence of a $K!$ behavior had been anticipated by Baker *et al.*,[5] who used the Padé-Borel method to sum the series and obtain values for the critical exponents for Ising-like systems ($n=1$). The more complete information now available allows us to apply a more precise method. Defining the Borel transform of $A(g)$ by

$$B(g)=\sum_K (A_K/K!)g^K, \qquad (3)$$

we can determine from Eq. (2) the position and the structure of the nearest singularity of $B(g)$. Under the assumption that $B(g)$ is analytic in a cut plane, it is possible to map the cut plane on a circle and obtain a convergent expansion[10] for $B(g)$—and therefore $A(g)$—so that

$$A(g)=\int_0^\infty e^{-t}B(tg)\,dt. \qquad (4)$$

As shall be explained below, this method can be further refined with use of the known coefficient b of Eq. (2). Our best estimates for the critical exponents are presented in the following table:

n	1	2	3	0
g^*	1.414± 0.003	1.405± 0.002	1.391± 0.001	1.4170± 0.0045
γ	1.2402± 0.0009	1.3160± 0.0012	1.3866± 0.0012	1.1615± 0.0011
η	0.0315± 0.0025	0.0335± 0.0025	0.0340± 0.0025	0.0260± 0.0030
ν	0.6300± 0.0008	0.6693± 0.0010	0.7054± 0.0011	0.5880± 0.0010
ω	0.782± 0.010	0.778± 0.008	0.779± 0.006	0.790± 0.015
Δ_1, $\omega\nu$	0.493± 0.007	0.521± 0.006	0.550± 0.005	0.465± 0.010
β	0.325± 0.001	0.346± 0.001	0.3647± 0.0012	0.3020± 0.0013

Here, g^* is the fixed-point value of the renormalized coupling constant, normalized in such a way that $W(g)=-g+g^2+O(g^3)$, and γ and ν are the critical exponents which govern the behavior near the critical temperature T_c of the magnetic susceptibility χ and of the correlation length ξ, respectively, such that

$$\chi\sim|T-T_c|^{-\gamma}, \quad \xi\sim|T-T_c|^{-\nu}. \qquad (5)$$

The exponent η gives the large-distance behavior at T_c of the spin-spin correlation function $G(x)\sim x^{2-d-\eta}$, and ω governs the leading corrections to scaling.

Our results for the critical exponents in the case with $n=1$ are compatible with, but more precise than, those obtained in Ref. 5: $\gamma=1.2410\pm0.002$; $\eta=0.021\pm0.02$; $\nu=0.627\pm0.01$; $\omega=0.78\pm0.01$; and $\Delta_1=0.49\pm0.01$. Our results give also more precise values than the most recent high-temperature–series estimates[11] and are in agreement with them, even now for the exponent γ although the value is definitively lower than the central value given by high-temperature series. Our results are also in very good agreement with the most recent experimental results[12] (see Table II). We shall briefly sketch the method which we used to derive these results.[13]

The critical behavior of a second-order phase transition is governed[2] by the infrared-stable zero g^* of the renormalization-group function $W(g)$ of the $g[\vec{\varphi}^2(x)]^2$ field theory, which is defined through the renormalization constants $Z(g)$ for the field and $Z_{(4)}(g)$ for the vertex by

$$W(g)=(d-4)[\partial\ln g_0/\partial g]^{-1}, \quad g_0=gZ_{(4)}/Z^2. \qquad (6)$$

$Z_{(2)}$ being the renormalization constant for the $\vec{\varphi}^2$ insertion, the critical exponents η, ν, and γ are determined by the functions[2]

$$\begin{aligned}\eta(g)&=W(g)[d\ln Z(g)/dg],\\ \nu(g)&=[2+W(g)d\ln Z_{(2)}/dg-\eta(g)],\\ \gamma(g)&=\nu(g)[2-\eta(g)],\end{aligned} \qquad (7)$$

evaluated at $g=g^*$, i.e.,

$$\eta=\eta(g^*), \quad \nu=\nu(g^*), \quad \gamma=\gamma(g^*). \qquad (8)$$

Finally, the leading corrections to the scaling laws are governed by the exponent

$$\omega=W'(g^*). \qquad (9)$$

Let us denote by $A^{(i)}(g)$, for $i=1$–5, the functions $W(g)$, $W'(g)$, $\eta(g)$, $\nu(g)$, and $\gamma(g)$, respectively. Their perturbation expansion is

$$A^{(i)}(g)=\sum_K A_K^{(i)}g^K, \qquad (10)$$

where the coefficients $A_K^{(i)}$ have been calculated by Nickel[8] up to $K=7$ for $W(g)$ and up to $K=6$ for the other functions. It has recently been shown[7] that the asymptotic behavior of $A_K^{(i)}$ for large K is given by

$$A_K^{(i)}\sim K!(-a)^K K^{b(i)}C_{(i)}[1+O(1/K)], \qquad (11)$$

where the numerical values of a and b are as fol-

TABLE II. Comparison of our results (first column) with those of Baker *et al.* (second column; Ref. 5), experimental data (third column; Ref. 12), and high-temperature-series results (fourth and rightmost column; Ref. 11).

	n = 1			
γ	1.2402±0.0009	1.241±0.002	1.240±0.007 [a] 1.23 1.24 1.27 1.28 [b]	$1.250^{+0.003}_{-0.007}$ [e]
η	0.0315±0.0025	0.021±0.02	0.016±0.007 0.016±0.014 [a]	
ν	0.6300±0.0008	0.627±0.01	0.625±0.003 0.625±0.005 [a]	$0.638^{+0.002}_{-0.008}$ [e]
β	0.325 ±0.001	0.320±0.016	0.321 0.323 0.329 [b] 0.316±0.008 0.328±0.004 [c]	0.312±0.005 [f]
	n = 2			
ν	0.6693±0.0010		0.675±0.001 [d]	
	n = 3			
γ	1.3866±0.0012			$1.375^{+0.02}_{-0.01}$ [g] 1.405±0.02 [h] $1.42^{+0.02}_{-0.01}$ [i]
η	0.0340±0.0025			0.043±0.014 [g] 0.040±0.008 [h]
ν	0.7054±0.0011			$0.7025^{+0.010}_{-0.005}$ [g] 0.717±0.007 [h] 0.725±0.015 [i]

[a]Chang *et al.*, Ref. 12.
[b]Hocken and Moldover, Ref. 12.
[c]Greer, Ref. 12.
[d]Mueller *et al.*, Ref. 12; Greywall and Ahlers, Ref. 12.
[e]Camp *et al.*, Ref. 11.
[f]Domb, Ref. 11.
[g]Ritchie and Fischer, Ref. 11.
[h]Ferer *et al.*, Ref. 11.
[i]Camp and Van Dyke, Ref. 11.

lows:

$$a=0.147\,774\,22\ldots,$$

$$b_{(i)}=\begin{cases}5+\tfrac12 n \text{ for } \omega(g);\\ 3+\tfrac12 n \text{ for } W(g),\nu(g),\gamma(g);\\ 2+\tfrac12 n \text{ for } \eta(g).\end{cases} \qquad (12)$$

We introduce the generalized Borel transform $B^{(i)}(gt)$ of the function $A^{(i)}(g)$ by writing

$$A^{(i)}(g)=\int_0^\infty e^{-t}t^{b'}\sum_K \frac{A_K^{(i)}U^K}{\Gamma(K+b'+1)}\,dt \equiv \int_0^\infty e^{-t}t^{b'}B^{(i)}(U)\,dt, \qquad (13)$$

where b' will be varied in the neighborhood of b and where $U=gt$. The important point is that the behavior (11) for large K gives the location and the nature of the nearest singularity of the Borel transform $B^{(i)}(U)$. This singularity is located at $U=-1/a$, where $B^{(i)}(U)$ behaves as $(1+aU)^{b'-b-1}$, or as $\ln(1+aU)$ for $b'=b+1$. Assuming that $B^{(i)}(U)$ is analytic in the U-plane cut from $-1/a$ to $-\infty$, we can now map this cut plane into a circle in the x plane with

$$U=x(1-\tfrac14 ax)^{-2}=\sum_{n=1}^{\infty}U_n x^n, \qquad (14)$$

the integration in (13) running from $x=0$ to $x=4/a$. The Borel transform is now given in (13) through a *convergent* series in x, which leads to

$$A^{(i)}(g)=\sum_n B_n\{\int_0^\infty e^{-t}t^{b'}[x(t)]^n\,dt\}. \qquad (15)$$

We have first used this result to calculate the zero g^* of $W(g)$. We have done this by several methods. One method consists of calculating the zero g^* of $W(g)$ for different number of terms of the new expansion (15). Because the successive values of g^* seemed to converge very smoothly, we extrapolated them with use of Padé approximants. Another approach used was to write $W(g)$ as $-gF(g)$ or $-g+g^2F(g)$, and then apply our method on the $F(g)$. This latter method seems to converge faster than the former one and, without the use of any Padé extrapolation, yields precise values of g^* which are consistent with those obtained from the other methods. In each case, we have varied b' in a neighborhood of b in order to generate a function $B(U)$ with a weak singularity $[(1+aU)^{1/2},\ \ln(1+aU),\ (1+aU)^{-1/2}]$ at $U=-1/a$. This has given us a range of values for g^*, for which we have calculated the other quantities $A^{(i)}(g)$, again with application of the same methods as for $W(g)$.

We have made other checks like calculating exponents from the direct series and its inverse, and verified the scaling laws $2\beta=3\nu-\gamma$ and $\gamma=\nu(2-\eta)$ between the independently computed exponents. All the results that we have obtained are consistent with one another.

Finally, several advantages of our approach with respect to the Padé-Borel method are to be emphasized. First, it gives more stable and accurate values, as we have verified, for instance, on the calculation of the ground-state energy of the anharmonic oscillator where, for an intermediate coupling, the accuracy is, for six terms of the perturbation series, of the order of 10^{-2} for the Padé method, 10^{-3} for the Padé-Borel method, and 3×10^{-4} for our method. Second, to improve efficiently the rate of convergence of a series by means of the Padé-Borel method, the perturbation series has to alternate in sign. This condition is not crucial in our approach. This is particularly important in the calculation of critical exponents from φ^4 field theory in three dimensions, where Baker *et al.*[5] only used $1/\gamma(g)$ and $\eta_2(g)\equiv W(g)(d/dg)\ln[Z_{(2)}(g)]$, whose series alternate in sign, the other critical exponents being obtained by scaling relations. With our method, we could calculate all exponents independently.

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[1]K. G. Wilson, Phys. Rev. B 4, 3184 (1971).

[2]E. Brézin, J. C. Le Guillou, and J. Zinn-Justin, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and M. S. Green (Academic, New York, 1976), Vol. VI.

[3]K. G. Wilson and M. E. Fisher, Phys. Rev. Lett. 28, 240 (1972).

[4]G. Parisi, in Proceedings of the 1973 Cargèse Summer School, edited by E. Brézin and J. M. Charap (Gordon and Breach, New York, to be published).

[5]G. A. Baker, B. G. Nickel, M. S. Green, and P. I. Meiron, Phys. Rev. Lett. 36, 1351 (1976).

[6]L. N. Lipatov, Pis'ma Zh. Eksp. Teor. Fiz. 25, 116 (1977) [JETP Lett. (to be published)], Leningrad Institute of Nuclear Physics Reports No. 253 and No. 255, 1976 (to be published).

[7]E. Brézin, J. C. Le Guillou, and J. Zinn-Justin, Phys. Rev. D 15, 1544, 1558 (1977).

[8]B. G. Nickel, to be published. Our Table I gives numbers used here.

[9]P. G. de Gennes, Phys. Lett. 44A, 271 (1973); J. des Cloizeaux, Phys. Rev. A 10, 1665 (1974), and J. Phys. (Paris) 36, 281 (1975). See also D. S. McKenzie, Phys. Rep. 27C, 35 (1976).

[10]J. J. Loeffel, Centre d'Etudes Nucléaires de Saclay Report No. SACLAY-DPh-T/76-20 (unpublished).

[11]W. J. Camp, D. M. Saul, J. P. Van Dyke, and M. Wortis, Phys. Rev. B 14, 3990 (1976); W. J. Camp and J. P. Van Dyke, J. Phys. A 9, 731 (1976); C. Domb, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and M. S. Green (Academic, New York, 1974), Vol. 3; M. Ferer, M. A. Moore, and M. Wortis, Phys. Rev. B 4, 3954 (1971); D. S. Ritchie and M. E. Fisher, Phys. Rev. B 5, 2668 (1972).

[12]R. F. Chang, H. Burstyn, J. V. Sengers, and A. J. Bray, Phys. Rev. Lett. 37, 1481 (1976); R. Hocken and M. R. Moldover, Phys. Rev. Lett. 37, 29 (1975); S. C. Greer, Phys. Rev. A 14, 1770 (1976); K. H. Mueller, F. Pobell, and G. Ahlers, Phys. Rev. Lett. 34, 513 (1975); D. S. Greywall and G. Ahlers, Phys. Rev. A 7, 2145 (1973).

[13]Applying the same method for $d=2$ and $n=1$ (Ising model) we have found $g^*=1.85\pm0.07$, $\gamma=1.79\pm0.07$, $\eta=0.19\pm0.07$, $\nu=0.98\pm0.07$, and $\omega=1.1\pm0.3$. In this case the perturbation series has only been calculated (Ref. 8) up to order 5 for $W(g)$ and to order 4 for the other functions.

Ultrasonic Propagation and Structural Instabilities in Itinerant-Electron Ferromagnets

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An itinerant-electron model for ultrasonic propagation in a ferromagnetic metal, for both above and below the Curie temperature T_C, is developed within the mean-field approximation. The attenuation maxima of the longitudinal acoustic wave are shown to occur at slightly below T_C in agreement with the experimental observation on Ni. In addition, new possibilities of magnetically driven structural instabilities in metals are pointed out.

The ultrasonic method has been used extensively in studying ferromagnetic substances.[1] Most of the previous theoretical treatments[2] of the ultrasonic propagation, however, are based on the localized-spin model for ferromagnetism. In this Letter I develop a simple itinerant-electron model for the ultrasonic behavior of a ferromagnetic metal.

In discussing the lattice vibrations of a metal, either magnetic or nonmagnetic, it is most important to consider the screening of the ion-ion interaction by the conduction electrons.[3] I pursue