The ballooning-mode code is used to compute critical volume average β versus q on axis for near-circular equilibria and fixed aspect ratio, where $\beta = 8\pi \int p\,dv/B_T^2$. Here B_T is the vacuum toroidal field. In Fig. 2, the vertical dashed curve at $q=1$ indicates the threshold for interchange stability. The solid curve indicates the ballooning-mode threshold for our theory. A dashed curve that varies as q^{-2} is also plotted. We see that the curves coincide for large q. For fixed q, we have found also that critical β varies linearly as the aspect ratio.

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Dynamic Scaling near Bicritical Points

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We calculate by renormalized field theory, up to two-loop order, the bicritical dynamic exponents of stochastic models appropriate for describing spin-flop bicritical points. In the presence of reversible mode-coupling terms, two-loop contributions establish bicritical dynamic scaling in the restricted sense and invalidate recent predictions based on mode-coupling arguments. In case of a purely relaxational model, total bicritical dynamic scaling is found to second order in $\epsilon = 4-d$.

The verification of the dynamic scaling hypothesis[1] for various models was one of the important results achieved by the renormalization-group approach to critical dynamics.[2] Although in some cases there still exist difficulties in interpreting the results of an ϵ expansion near an ordinary critical point,[3] one is led to employ this successful method also in investigating dynamic scaling near *multicritical* points. Siggia and Nelson[4] have recently studied tricritical dynamics to linear order in $\epsilon = 4-d$; in case of He^3-He^4 mixtures they find ambiguities similar to those appearing in model C of Halperin, Hohenberg, and Ma (HHM).[3]

In this Letter, we report the first renormalization-group analysis of *bicritical*[5] dynamics. We formulate stochastic bicritical models in terms of the Lagrangian version of the Martin-Siggia-Rose theory,[6] as given previously by one of us,[7] and employ the field-theoretic renormalization-group approach of Bausch, Janssen, and Wagner[8] combined with the minimal subtraction procedure.[9] In the presence of reversible mode-coupling terms we find that bicritical dynamics must be treated at least up to two-loop order in order to avoid a spurious breakdown of scaling at $O(\epsilon = 4-d)$ and to work out the leading deviations from conventional theory. These results invalidate a recent prediction[10] given on the basis of mode-coupling and scaling arguments for the $n=3$ spin-flop bicritical point in uniaxial antiferromagnets having rotational symmetry around the anisotropy axis. Our analysis will show that bicritical dynamic scaling, at least in the restricted sense, is indeed valid to two-loop order in several cases including the case considered by Huber and Raghavan.[10] Furthermore, our results provide additional motivation for reexamining tricritical dynamics[4] by a two-loop analysis.

We start from the bicritical Hamiltonian

$$H=\tfrac{1}{2}\int d^dx[r_{\parallel}\vec{\sigma}^2+(\nabla\cdot\vec{\sigma})^2+r_{\perp}\vec{s}^2+(\nabla\cdot\vec{s})^2+m^2+hm+U(\vec{\sigma}^2)^2+2W\vec{\sigma}^2\vec{s}^2+V(\vec{s}^2)^2+Am\vec{\sigma}^2+Bm\vec{s}^2], \tag{1}$$

which is identical with that of Kosterlitz, Nelson, and Fisher[5] except for the additional terms containing the one-component nonordering density m. In a dynamic theory these terms must be treated explicitly provided that m is a conserved density (e.g., the z component of the magnetization of an $n_{\parallel}+n_{\perp}=3$ component uniaxial antiferromagnet in an external homogeneous magnetic field h). $\vec{\sigma}(\vec{x})$ and $\vec{s}(\vec{x})$ denote $n_{\parallel}$-component and $n_{\perp}$-component order-parameter densities, respectively. Our *dynamic* model is defined by the action integral[7] $J=\int dtL$ with the Lagrangian[11]

$$L=\int d^dx[\tilde{\sigma}_{\alpha}(L_{\sigma}\tilde{\sigma}_{\alpha}-\dot{\sigma}_{\alpha}-L_{\sigma}\delta H/\delta\sigma_{\alpha})+\tilde{s}_{\alpha}(L_s\tilde{s}_{\alpha}-\dot{s}_{\alpha}-L_{\alpha\beta}\delta H/\delta s_{\beta}+G_{\alpha\beta}s_{\beta}\delta H/\delta m)$$
$$+\tilde{m}(-L_m\nabla^2\tilde{m}-\dot{m}+L_m\nabla^2\delta H/\delta m-G_{\alpha\beta}s_{\beta}\delta H/\delta s_{\alpha})] \tag{2}$$

(summation over repeated indices is implied). The variables $\tilde{\vec{\sigma}}(\vec{x}t)$, $\tilde{\vec{s}}(\vec{x}t)$, and $\tilde{m}(\vec{x}t)$ are response fields. The L_i's are kinetic coefficients, $L_{\alpha\beta}=L_s\delta_{\alpha\beta}+F_{\alpha\beta}$; $G_{\alpha\beta}$ and $F_{\alpha\beta}$ are antisymmetric mode-coupling matrices. All relevant couplings are retained in (2) as can be seen from dimensional arguments applied to J, in complete analogy to the static case.

We shall examine the following cases: (I) $n_{\parallel}=1$, $n_{\perp}=2$, with $G_{12}=G$, $F_{12}=F$; (II) $n_{\parallel}$ and $n_{\perp}$ arbitrary but m *not* conserved. Applications are the spin-flop bicritical points in MnF_2 (case I) and in $GdAlO_3$ (case II with $n_{\parallel}=n_{\perp}=1$) and possibly tetracritical points in higher-component systems exhibiting displacive transitions[5] (case II with $n_{\parallel}+n_{\perp}\geq 4$).

We proceed according to renormalized field theory[8] by introducing renormalized fields $\varphi_j=Z_j^{-1/2}j$ ($j=\vec{\sigma},\vec{s},m,\tilde{\vec{\sigma}},\tilde{\vec{s}},\tilde{m}$) and a set p_i of renormalized dimensionless parameters. We need in particular the ratios $\rho_{\sigma}=\lambda_{\sigma}/\lambda_m$, and $\rho_s=\lambda_s/\lambda_m$ with $\lambda_i=Z_{\lambda_i}L_i$, the static parameters $a=A/4\pi\mu^{\epsilon/2}Z_a$, $b=B/4\pi\mu^{\epsilon/2}Z_b$, and the mode-coupling constants $g=G/4\pi\mu^{\epsilon/2}\lambda_mZ_g$, $f=F/4\pi\mu^{\epsilon/2}\lambda_mZ_f$; here μ^{-1} is the usual parameter defining a length scale in the renormalized theory. With the functions

$$\gamma_{\varphi_j}=(\mu\partial_{\mu}\ln Z_{\varphi_j})_0,\quad \zeta=(\mu\partial_{\mu}\ln Z_{\lambda_m})_0,\quad \beta_{p_i}=(\mu\partial_{\mu}p_i)_0$$

(the subindex 0 means differentiation at fixed unrenormalized parameters), the renormalization-group equation for renormalized vertex functions $\Gamma_{\{n_j\}}$ reads

$$\{\mu\partial_{\mu}+\zeta\lambda_m\partial_{\lambda_m}+\textstyle\sum_i\beta_{p_i}\partial_{p_i}+\sum_{\alpha\beta}\tau_{\alpha}\kappa_{\alpha\beta}\partial_{\tau_{\beta}}-\frac{1}{2}\sum_j n_j\gamma_{\varphi_j}\}\Gamma_{\{n_j\}}=0. \tag{3}$$

The integers n_j count the number of fields φ_j associated with the vertex function under consideration. τ_1 and τ_2 are renormalized linear measures of the deviations from the bicritical point in the h-T plane. The 2×2 matrix $\kappa_{\alpha\beta}$ as defined by $\sum_{\alpha}\tau_{\alpha}\kappa_{\alpha\beta}=(\mu\partial_{\mu}\tau_{\beta})_0$ is diagonal if τ_1, τ_2 are scaling variables; hence its eigenvalues at the fixed point are $2-\nu^{-1}$ and $2-\phi\nu^{-1}$ with ϕ being the crossover exponent.[5]

In determining the Z factors we have taken advantage of the minimal renormalization procedure.[9] As pointed out by De Dominicis and Peliti,[12] allows one to renormalize statics and dynamics separately. Furthermore, in this procedure dissipation-fluctuation theorems result in general relations between the Z factors, whose validity is not restricted to fixed points and which provide an enormous simplification in higher-order computations. For our models such relations are $Z_g^2=Z_m$, $Z_{\lambda_{\sigma}}^2=Z_{\tilde{\sigma}}/Z_{\sigma}$ (model I), and $Z_{\lambda_{\sigma}}^2=Z_{\tilde{\sigma}}/Z_{\sigma}$, $Z_{\lambda_s}^2=Z_{\tilde{s}}/Z_s$ (model II).

Model I ($n_\parallel = 1$, $n_\perp = 2$).—First we extend previous static results[5] by treating the explicit m couplings in (1). Because within statics m can be integrated out, there exist the following exact relations (for arbitrary $n_\parallel$ and $n_\perp$):

$$\beta_a = a[\kappa_{11} - (\epsilon + \gamma_{\varphi_m})/2] + b\kappa_{21}, \tag{4}$$

$$\beta_b = a\kappa_{12} + b[\kappa_{22} - (\epsilon + \gamma_{\varphi_m})/2]. \tag{5}$$

They imply, according to the two eigenvalues of $\kappa_{\alpha\beta}^*$, the fixed-point values $\gamma_{\varphi_m}^* = d - 2\nu^{-1}$ and $\gamma_{\varphi_m}^* = d - 2\varphi\nu^{-1}$. For the Heisenberg fixed point[5] the latter value corresponds to a stable fixed point with $a^*/b^* = -n_\perp/n_\parallel = -2$ (exact) and $a^{*2} = n_\perp(2\phi\nu^{-1} - d)/(n_\parallel^2 + n_\parallel n_\perp) + O(\epsilon^3)$. The corresponding correction exponents are $\omega_1 = (\phi - 1)/\nu$ and $\omega_2 = 2\phi\nu^{-1} - d$ which again are exact results.

For the *dynamics* of model I it is convenient to introduce the complex renormalized parameter $\rho = \rho_s + if$. To one-loop order we obtain the dynamic β functions

$$\beta_\rho = -\rho(2a^2 + b^2 - g^2/\rho_s) + 2(a\rho + ig)^2/(1+\rho), \tag{6}$$

$$\beta_{\rho_\sigma} = -\rho_\sigma[\zeta + \tfrac{1}{2}(\gamma_{\varphi_\sigma} - \gamma_{\varphi_{\tilde\sigma}})] = -\rho_\sigma(2a^2 + b^2 - g^2/\rho_s) + 2(b\rho_\sigma)^2/(1+\rho_\sigma), \tag{7}$$

$$\beta_g = -g(\epsilon + 2\zeta + \gamma_{\varphi_m})/2 = -g(\epsilon/2 + a^2 + b^2/2 - g^2/\rho_s). \tag{8}$$

The first parts of Eqs. (7) and (8) are exact as follows from $Z_{\lambda_\sigma}^2 = Z_{\tilde\sigma}/Z_\sigma$ and $Z_g^2 = Z_m$. The zeros of (6)-(8) yield the stable fixed point $\rho^* = 1.496 \pm i1.585$, $\rho_\sigma^* = 0$, $g^* = \pm(144\pi^2\epsilon\rho_s^*/11)^{1/2}$, with a positive (negative) sign of f^* if the product a^*g^* is taken negative (positive). The vanishing of ρ_σ^*/ρ_s^* leads to two different dynamic exponents z_σ and z_s for the "parallel" and "perpendicular" order-parameter correlation function and implies a breakdown of bicritical dynamic scaling. To linear order in ϵ we find $z_\sigma = 2$, $z_s = \phi/\nu = 2 - 2\epsilon/11$, and the transient exponents $\omega_\rho = (0.299 \pm i0.283)\epsilon$, $\omega_{\rho_\sigma} = 2\epsilon/11$, and $\omega_g = 0.928\epsilon$. These results seem to confirm the recent prediction[10] based on mode-coupling arguments. Naturally at this point the question arises as to the importance of "corrections of order ϵ^2."

We have found, however, that the two-loop contributions to the β functions cannot be considered as "corrections" but establish a new stable fixed point *for arbitrarily small* ϵ. This unusual feature results from the singular two-loop term

$$-\rho\ln(\rho_\sigma/\rho_s)\{[2w(1+\rho) - ab]^2 + igb[w(1+\rho) - ab]\}/(1+\rho)^2, \tag{9}$$

which complements the right-hand side of (6), apart from additional (but well-behaved) contributions. The logarithm in (9) no longer permits a finite value of ρ_s^*, i.e., a vanishing of ρ_σ^*/ρ_s^*. Instead we find at two-loop order the *stable* dynamic fixed point

$$\rho^* = 0, \quad \rho_\sigma^* = 0, \quad g^{*2} = 144\pi^2\epsilon/11, \tag{10}$$

with the *finite* ratio $\rho_\sigma^*/\rho_s^* \sim \exp(-198/\epsilon)$. This implies $z_\sigma = z_s \equiv z$ and therefore restores bicritical dynamic scaling in the restricted sense. We have a violation of extended dynamic scaling due to the different exponent $z_m = 2 + \zeta^*$ governing the m–m correlation function. z_σ is related to z_m via $z_\sigma = z_m + \beta_{\rho_\sigma}^*/\rho_\sigma^*$. From (7) and (8) we get $z_\sigma = 2 + \frac{1}{2}(\gamma_{\varphi_{\tilde\sigma}}^* - \gamma_{\varphi_\sigma}^*)$ and $z_m = \phi/\nu$ (exact). To two-loop order we find

$$z = 2 + c\eta + O(\lambda_\sigma^*/\lambda_s^*) \tag{11}$$

with $c = \frac{18}{5}\ln\frac{4}{3} - 1 = 0.0357$.

Besides model C of HHM[3]—and perhaps the tricritical model of Siggia and Nelson[4]—our bicritical model I represents a novel example where a nonanalytic ϵ dependence appears[13] and two-loop contributions cause a qualitative change of $O(\epsilon)$ results for arbitrarily small ϵ. Although we have some confidence in the relevance of our two-loop results at least in a qualitative sense, we cannot rule out the possibility that higher-loop terms even restore *extended* bicritical scaling and also drive the numerical value of ρ_σ^*/ρ_s^* to a less fantastic order of magnitude.

Model II.—We assume m to be a *nonconserved* density. In this case the critical dynamics of the order-parameter components is not affected by the m variable. Therefore we drop the corresponding terms in (1)–(3) from the outset and deal with a purely relaxational model for the $\vec{s}$ and $\vec{\sigma}$ fields. The dynamic parameter of interest is the ratio $\lambda \equiv \lambda_\sigma/\lambda_s$ of the renormalized kinetic coefficients. The leading contributions to the corresponding β function $\beta_\lambda = (\mu\partial_\mu\lambda)_0$ arise in two-loop order and are given by

$$\beta_\lambda = (\lambda/18)\{[(n_\perp+2)v^2-(n_\parallel+2)u^2](6\ln\tfrac{4}{3}-1)+n_\parallel w^2[2\ln(\alpha^2/\beta)+4\lambda\ln(2\alpha/\beta)-1] - n_\perp w^2[2\ln(\alpha^2/\gamma\lambda)+4\lambda^{-1}\ln(2\alpha/\gamma)-1]\}, \quad (12)$$

with $\alpha\equiv 1+\lambda$, $\beta\equiv 1+2\lambda$, $\gamma\equiv 2+\lambda$.

For the static Heisenberg fixed point[5] one obtains $\lambda^*=1$, independent of $n_\parallel$ and $n_\perp$. For the biconical fixed point[5] we consider $n_\parallel=1$ and $n_\perp = n-1$. The corresponding values $\lambda_B^*(n)$ are given in Table I which complements the static exponents table of Kosterlitz, Nelson, and Fisher.[5] Table I also contains the dynamic exponent $z\equiv z_s = z_\sigma$ for the biconical and Heisenberg case (with $n_\parallel=1$ and $n_\perp=n-1$). In the latter case the correction exponent is $\omega_\lambda = 2n\epsilon^2(n+8)^{-2}\ln\frac{4}{3}$.

In conclusion, bicritical dynamic scaling is obeyed both for model I (in the restricted sense) and for model II (as long as static scaling holds). Thus in the asymptotic scaling region there exist characteristic frequencies for the "parallel" and "perpendicular" order-parameter correlation function governed by a common dynamic exponent z and common transient exponents ω_i. It would be interesting to test this prediction experimentally by inelastic neutron scattering near the spin-flop bicritical points in MnF_2 ($z=2.0007$) and in $GdAlO_3$ ($z=2.015$, $\omega_\lambda=0.01$).

Finally we briefly touch on the question as to the experimental accessibility of the asymptotic dynamic scaling region. Although our theory cannot answer this question quantitatively we suspect that, according to the extremely small ratio ρ_σ^*/ρ_s^*, an $n=3$ system like MnF_2 may show an effective behavior corresponding to the $O(\epsilon)$ fixed point ($\rho_\sigma^*/\rho_s^*=0$), i.e., with two different effective exponents $z_\sigma=2.00$ and $z_s=z_m=\phi/\nu=1.78$, even very close to the bicritical point. We hope that future experiments in MnF_2 are sufficiently accurate to answer this question and eventually show an ultimate crossover to the asymptotic dynamic exponent (11).

TABLE I. Dynamic diconical (z_B) and Heisenberg (z_H) exponents for model II evaluated to order ϵ^2 at $\epsilon=1$.

n	$\lambda_B^*(n)$	$z_B(n)$	$z_H(n)$
1	1.1145	2.0135	2.0135
2	1.0000	2.0135	2.0145
3	1.0241	2.0149	2.0150
4	1.0000	2.0151	2.0151
5	1.0205	2.0151	2.0150
6	1.0846	2.0149	2.0148
7	1.2131	2.0147	2.0145
9	2.1588	2.0141	2.0138
10	6.0005	2.0138	2.0135
11	$\cdots$	2.0135	2.0131
13	0.0090	2.0129	2.0124
15	0.1271	2.0122	2.0117

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