Theory of Ballooning Modes in Tokamaks with Finite Shear

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We studied ballooning instabilities in tokamaks of arbitrary cross section and finite shear. These azimuthally localized, ideal magnetohydrodynamic modes have large toroidal-mode numbers, but finite variation along the field and across the flux surfaces. Stability is determined by solving a second-order ordinary differential equation on each flux surface, subject to the proper boundary conditions. Qualitative agreement is achieved with the Princeton PEST stability code.

The economics of conceptual tokamak reactors improves significantly for β as large as 10%, where β is the ratio of plasma pressure to magnetic pressure. The achievable β is likely to be determined by "ballooning modes." These are magnetoydrodynamic (MHD) instabilities, analogous to the aneurisms which develop at weak spots in a pressurized elastic container. In plasmas, these modes are driven by the interaction of the plasma pressure gradient with local regions of unfavorable magnetic curvature; they cause the plasma to bulge out in these regions. Since they bend the field lines, they are partially stabilized by magnetic tension and thus require some minimum pressure gradient before instability develops.

A number of attemps to describe ballooning modes for tokamaks appear in the literature[1-7] for special geometries, including numerical codes for general cross section.[8] However, these computations do not take into account the effect of shear, which must be included to describe adequately the ballooning stability threshold.

Interchange stability criteria have been derived for modes that are concentrated near a magnetic surface.[9] This type of mode is very ineffective in tapping the free energy associated with local normal field-line curvature, since its displacements lie predominantly in the magnetic surfaces. On the other hand, azimuthally localized modes have displacements that are primarily normal to the flux surfaces. Thus they can efficiently maximize local destabilizing energy associated with normal curvature, with a minimum of field-line bending. In axially symmetric tokamaks, azimuthally localized modes are associated with a large toroidal-mode number, n. That limit is the subject of this Letter.

A theory of such modes has several advantages over large two-dimensional stability codes such as PEST and ERATO.[10] For high n, the problem of stability is reduced to solving single second-order ordinary differential equation on each surface independently. This additional simplicity aids in our understanding of the modes and may help us to interpret many features of larger codes at a fraction of the computation time. In fact, this ballooning-mode theory is complementary to such codes since they are unable to calculate stability for high n values. This theory also enables us to examine complex geometries, such as doublets, and may allow us to include additional physical effects, such as resistivity and inertia.

In this Letter, we develop the equations for the ballooning mode from the energy principle.[11] The change in potential energy (δW) due to a virtual displacement $\vec{\xi}$ is given by an energy functional,

$$\delta W=\tfrac{1}{2}\int d\tau\,[B^{-2}(\vec{Q}\times\vec{B})^2+B^{-2}(\vec{Q}\cdot\vec{B}-\vec{\xi}\cdot\nabla p)^2+\gamma p(\nabla\cdot\vec{\xi})^2-(B^{-2}\vec{J}\cdot\vec{B})(\vec{\xi}\times\vec{B}\cdot\vec{Q})-2(\vec{\xi}\cdot\nabla p)(\vec{\xi}\cdot\vec{\kappa})]. \quad (1)$$

Here $\vec{Q}=\nabla\times(\vec{\xi}\times\vec{B})$ is the perturbed magnetic field. Equilibrium quantities are the pressure p, the current density $\vec{J}$, the magnetic field $\vec{B}$, and the field curvature $\vec{\kappa}$. The terms in Eq. (1) may be described as follows[12]: The first three stabilizing terms represent the work done in bending and compressing the field lines and compressing the plasma. The last two terms may be destabilizing. The term involving $\vec{J}\cdot\vec{B}$ drives the kink instability and plays a small role in interchanges. The last term, which drives a ballooning modes and interchanges, is the interaction of the pressure gradient and the magnetic-field curvature.

We use the Hamada coordinates (V,θ,ζ) with unit Jacobian to analyze Eq. (1). In this system V is the volume enclosed by a magnetic-flux surface; θ and ζ are poloidal- and toroidal-angle-like variables, which increase by unity in going around the torus the short and long ways, respectively. The magnetic field is given by $\vec{B}=\nabla V\times(\psi'\nabla\theta-\chi'\nabla\zeta)$, so that the derivative along the field line is $\vec{B}\cdot\nabla=\chi'(\partial/\partial\theta+q\partial/\partial\zeta)$ with the safety factor q defined by $q\equiv\psi'/\chi'$. The coordinate ζ is ignorable in axial symmetry. The quantities ψ and χ are the toroidal and poloidal magnetic fluxes, and are functions of V only. Primes indicate differentiation with respect to V.

The curvature $\vec{\kappa}$ may be written as

$$\vec{\kappa}=\kappa_v\chi'\nabla V+\kappa_s(\psi'\nabla\theta-\chi'\nabla\zeta)$$
$$=\tfrac{1}{2}\vec{B}\times[\nabla(2p+B^2)\times\vec{B}]/B^4. \tag{2}$$

Then $\kappa_s=(\vec{\kappa}\cdot\vec{B}\times\nabla V)/B^2$ and $\kappa_v=-[\vec{\kappa}\cdot\vec{B}\times(\psi'\nabla\theta-\chi'\nabla\zeta)]/\chi'B^2$ so that they are proportional to the geodesic and the normal curvature. For equilibria with shear, δW is minimized by displacements which are divergence-free.[9] Then, only components of $\vec{\xi}$ perpendicular to $\vec{B}$ enter δW, and we write

$$\vec{\xi}_\perp=[\xi_v(\psi'\nabla\theta-\chi'\nabla\zeta)\times\vec{B}+\xi_s\vec{B}\times\nabla V]/B^2. \tag{3}$$

Here ξ_v is the component of $\vec{\xi}$ normal to the flux surface, and ξ_s is the component in the surface, but normal to $\vec{B}$.

To create a model for the ballooning mode, we consider displacements with large variation, or order n, in V, θ, and ζ. Then the dominant terms in δW are stabilizing unless gradients along the field line are finite and $\nabla\cdot\vec{\xi}_\perp^{(0)}=0$ to lowest order. To next order in powers of $1/n$, minimization with respect to $\xi_s^{(1)}$ eliminates the field-line–compression term. Since $\vec{\xi}_\perp^{(0)}$ is incompressible, we can introduce a stream function φ, such that $\xi_v^{(0)}=\chi'^{-1}\partial\varphi/\partial\zeta$ and $\xi_s^{(0)}=\partial\varphi/\partial V$. Then we can show that the term driving the kink instability is an exact differential which integrates to zero to lowest order in $1/n$, where $\xi_v^{(0)}$ vanishes at the plasma boundary.

Since ζ is ignorable we take displacements of the form $\exp(-2\pi in\zeta)$. Then the derivative along the field line is $\vec{B}\cdot\nabla=\chi'(\partial/\partial\theta-2\pi inq)$. This suggests a change of dependent variable $F\equiv\exp[2\pi in(\zeta-q\theta)]\xi_v^{(0)}$. Then $\vec{B}\cdot\nabla\xi_v^{(0)}$ being finite translates to $\partial F/\partial\theta$ being finite. Results of PEST calculations indicate that the most unstable modes have radially extended channels, i.e., $\partial F/\partial V\ll nq'F$. In this approximation, δW reduces to $\delta W_{\min}=\frac{1}{2}\int dV\chi'^2\delta W(V)$, where

$$\delta W(V)=\oint\left\{\frac{\chi'^2|\nabla\alpha^2}{B^2}\left|\frac{\partial F}{\partial\theta}\right|^2-\frac{2p'}{\chi'}(\kappa_v-q'\theta\kappa_s)|F|^2\right\}d\theta, \tag{4}$$

Here $\alpha=\zeta-q\theta$ is constant along a field line. From Eq. (4), we see that V appears as a parameter only. Hence the effect of having high n and finite normal derivative of F is to decouple the stability analysis from surface to surface.

We normalize this functional, defining

$$H=1=\oint(\chi'^2|\nabla\alpha|^2/B^2)|F|^2, \tag{5}$$

and minimize $\lambda=\delta W/H$. Then stability is determined by the sign of λ. Minimization of $\delta W/H$ leads to the interior Euler equation,

$$\frac{\partial}{\partial\theta}\left(\frac{\chi'^2|\nabla\alpha|^2}{B^2}\frac{\partial F}{\partial\theta}\right)+\left[\lambda\frac{\chi'^2|\nabla\alpha|^2}{B^2}+\frac{2p'}{\chi'}(\kappa_v-q'\theta\kappa_s)\right]F=0. \tag{6}$$

Substitution into Eq. (4) yields

$$\delta W(V)=\frac{\chi'^2|\nabla\alpha|^2}{2B^2}\left(F^*\frac{\partial F}{\partial\theta}+F\frac{\partial F^*}{\partial\theta}\right)\Bigg|_{\theta_0}^{\theta_0+1}+\lambda. \tag{7}$$

Hence we have marginal stability if the right-hand side of Eq. (7) vanishes for displacements that satisfy Eq. (6).

In systems with finite shear, Eq. (6) has coefficients which are nonperiodic in θ. This is be-cause we have neglected the quantity $\partial F/\partial V$. However, from the periodicity of ξ_v we find that F satisfies the condition

$$F(V,\theta+1)=F(V,\theta)\exp(-2\pi inq). \tag{8}$$

Evidently, if $\partial F/\partial V$ is small for one value of θ, it is large when θ increases by unity. Thus the derivative can be small only within an interval of length ≤ 1. To make the length unity, we impose the constraint that F vanish at the endpoints. To define this approximation fully, we must specify the endpoints and the origin of the θ coordinate. In general, these can be regarded as variational parameters and used to search for the most unstable mode.

In systems with up-down symmetry, the most unstable mode has the origin of θ on the outside of the torus and nulls of F at $\theta=\pm\frac{1}{2}$, on the inner side. The most unstable mode has no other nodes and is even in θ, $F(-\theta)=F^*(\theta)$.

The boundary conditions $F(\pm\frac{1}{2})=0$, while derived from periodicity conditions, do not imply that $F(\theta)$ is periodic; it is not. This is clear from Eq. (8) and the fact that $\partial F/\partial V$ is small only in a finite interval in θ. There is no discontinuity, but a nonanalyticity, at the endpoints of the θ interval. This nonanalyticity is associated with the neglect of $\partial F/\partial V$ and can be resolved only by its inclusion in higher order. In any case, our calculation is a minimization of δW within a chosen class of perturbations. It thus yields a significant and useful sufficient condition for instability. A similar structure has been noted by Coppi and Rewoldt[13] for topologically similar kinetic modes, but it is noted that surfaces with rational safety factor q play no special role in our theory.

We have applied this theory to a set of nearly circular, fixed–aspect-ratio, numerical equilibria obtained by varying poloidal β_p and vacuum toroidal field B_T. For these equilibria, Eq. (6) is evaluated on each of eighteen poloidal flux surfaces. Without shear, Eq. (6) is Hill's equation and hence exhibits solutions similar to Mathieu functions. The lowest eigenvalue for each surface is sought and plotted versus flux-surface number in Fig. 1 for cases with and without shear. The figure illustrates shear stabilization. Shear plays a dual role in our stability theory; it provides stabilization due to field-line bending in the $|\nabla\alpha|^2$ term, and is stabilizing if geodesic displacements of field lines are in the direction of increasing magnetic pressure.

The same equilibria have been tested by the Princeton PEST MHD stability code and the marginally stable β value has been calculated. The balloon code has consistently found critical β's from 10% to 20% below those of PEST. The discrepancy between marginal-stability values of β indicates the complementary nature of the codes. The PEST code was run for $n=3$, and it is difficult to obtain numerically accurate results for longer values of n. On the other hand, our theory is strictly accurate only for very large n. Corrections could be calculated arising from such effects as the finite variation of $\partial F/\partial V$, coupled with the variation of $\lambda(V)$. For the cases compared, the stability predictions of the PEST code and those of the ballooning-mode theory qualitatively agree. The computation time of the ballooning-mode theory is less than 10% of that of PEST.

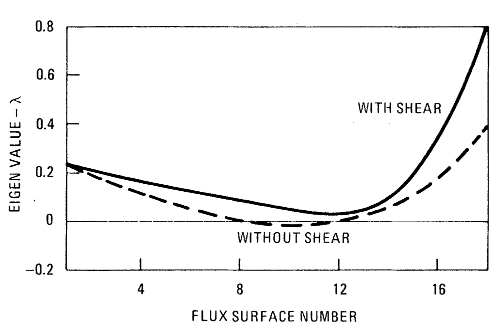

FIG. 1. λ vs poloidal flux-surface number with and without shear in Eq. (6). The aspect ratio is 5, the major radius is 150 cm, the total current is 190 kA, and B_T is 12 kG.

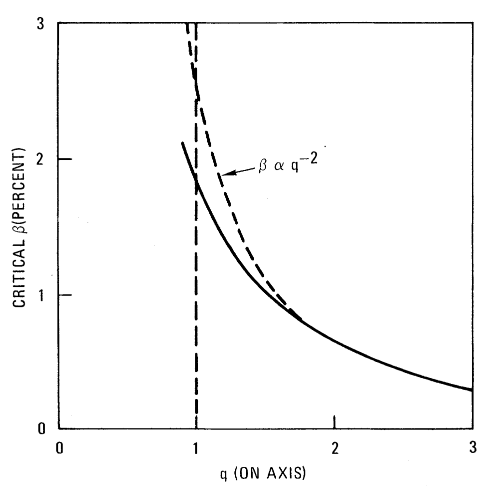

FIG. 2. Critical β vs q on axis. The vertical dashed curve is threshold for interchange stability and the solid curve is that for ballooning instability. Stability is below and to the right. B_T is varied from 8 to 35 kG.

The ballooning-mode code is used to compute critical volume average β versus q on axis for near-circular equilibria and fixed aspect ratio, where $\beta = 8\pi \int p\, dv/B_T^2$. Here B_T is the vacuum toroidal field. In Fig. 2, the vertical dashed curve at $q=1$ indicates the threshold for interchange stability. The solid curve indicates the ballooning-mode threshold for our theory. A dashed curve that varies as q^{-2} is also plotted. We see that the curves coincide for large q. For fixed q, we have found also that critical β varies linearly as the aspect ratio.

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Dynamic Scaling near Bicritical Points

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We calculate by renormalized field theory, up to two-loop order, the bicritical dynamic exponents of stochastic models appropriate for describing spin-flop bicritical points. In the presence of reversible mode-coupling terms, two-loop contributions establish bicritical dynamic scaling in the restricted sense and invalidate recent predictions based on mode-coupling arguments. In case of a purely relaxational model, total bicritical dynamic scaling is found to second order in $\epsilon = 4-d$.

The verification of the dynamic scaling hypothesis[1] for various models was one of the important results achieved by the renormalization-group approach to critical dynamics.[2] Although in some cases there still exist difficulties in interpreting the results of an ϵ expansion near an ordinary critical point,[3] one is led to employ this successful method also in investigating dynamic scaling near *multicritical* points. Siggia and Nelson[4] have recently studied tricritical dynamics to linear order in $\epsilon = 4-d$; in case of He^3-He^4 mixtures they find ambiguities similar to those appearing in model C of Halperin, Hohenberg, and Ma (HHM).[3]