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Connections between the Critical Behavior of the Planar Model and That of the Eight-Vertex Model

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Lines of critical points which emerge from the two-dimensional planar model's multicritical point are analyzed. These lines seem to be in the same universality class as both the Ashkin-Teller and the eight-vertex (8V) models' critical lines, with the multicritical point being isomorphic with the 8V model at both the points $\tanh 2\lambda = \sqrt{2}/2$ and -1. Expansions of critical indices about these values of λ are performed. If this analysis is right, then this work permits the evaluation of a whole class of interesting correlation functions at the critical point of the F model and that of the four-state Potts model.

In a recent publication, Jose, Kadanoff, Kirkpatrick, and Nelson[1] described an excitation model which in appropriate limits reduced to the Villain[2] form of the planar model and to the p-state planar Potts model. For $p=4$, the latter is the Ashkin-Teller (AT) model. They showed that this excitation model produced the Kosterlitz-Thouless[3,4] multicritical point (MCP) for the planar model and identified six lines of continuously varying critical behavior emerging from this multicritical point. In this Letter, we argue that one of these lines (which is isomorphic to three others) is, in fact, in the same universality class as the AT and eight-vertex (8V) models.

The excitation model has two kinds of integer quantum numbers: $m(\vec{R})$, which describes vortices at the dual-lattice sites, $\vec{R}$; and $n(\vec{r})$, which describes the breaking of the planar model's rotational symmetry. The excitation Hamiltonian, $H[n,m]$, is infinite if the sum over sites of $n(\vec{r})$ or $m(\vec{R})$ does not vanish. Otherwise, $H=H_0+V$, with[5]

$$H_0 = -\ln y_p \sum_{\vec{r}} [n(\vec{r})]^2 - \ln y_0 \sum_{\vec{R}} [m(\vec{R})]^2,$$

$$V[n,m] = -\tfrac{1}{2}K_n \sum_{\vec{r}\,\vec{r}\,'} n(\vec{r})V_1(\vec{r}-\vec{r}\,')n(\vec{r}\,') - \tfrac{1}{2}K_m \sum_{\vec{R}\vec{R}'} m(\vec{R})V_1(\vec{R}-\vec{R}')m(\vec{R}') - ip\sum_{\vec{r}\vec{R}} n(\vec{r})V_2(\vec{r}-\vec{R})m(\vec{R}). \tag{1}$$

This model has long-range interactions since, for large separations,

$$V_1(\vec{r}) + i\,V_2(\vec{r}) \to \ln(x+iy). \tag{2}$$

In this paper, we limit ourselves to $p=4$, which is the symmetry breaking appropriate to the AT model. In fact, the model reduces to the AT model in the AT limit:

$$K_n = 16/K_m; \quad y_0 = y_p = 1. \tag{3}$$

On the other hand, the excitation model is exactly solvable when the y's go to zero and has a multicritical behavior at the MCP

$$K_n = K_m = 4; \quad y_0 = y_p = 0. \tag{4}$$

Furthermore, this model seems to have a critical line:

$$K_n = K_m = K; \quad y_0 = y_p = H/2\pi > 0, \tag{5}$$

where K depends upon H. Our strategy is to expand about the point (4) on the line (5) and to use identifications built from the AT limit (3) to understand what we are seeing.

To implement this strategy calculate the partition function Z and the correlation function

$$\langle O[N,M]\rangle = \sum_{n,m} Z^{-1}\exp(-H_0[n,m]-V[N+n,M+m]). \quad (6)$$

Here $4N(\vec{r})$ and $M(\vec{R})$ are sets of integers which we choose to be nonvanishing on a few lattice sites $\vec{r}_j$ and $\vec{R}_k$. We then define local operators, $O_N(\vec{r})$ and $\tilde{O}_M(\vec{R})$, by writing $O[N,M]$ as a product:

$$O[N,M]=\Big[\prod_j O_{N(\vec{r}_j)}(\vec{r}_j)\Big]\Big[\prod_k \tilde{O}_{M(\vec{R}_k)}(\vec{R}_k)\Big]. \quad (7)$$

The connection with the AT model is that in the limit (3) we can exactly identify $O_N(\vec{r})$ in terms of the standard $S(\vec{r})$ and $\sigma(\vec{r})$ variables by the AT model as shown in line 1 of the Table I. Building upon this relation, we form the first three columns in the table, which describe the relationship between our most relevant operators and those of the AT model.

At the MCP (4), it is trivial to calculate correlations among all our operators. For example,

$$\langle O_N(\vec{r})O_{-N}(\vec{r}\,')\rangle = |\vec{r}-\vec{r}\,'|^{-2x_N}, \quad (8)$$

with the standard critical index x_N being given by $x_N=2N^2$ at the MCP. The fourth column of Table I gives the critical-index values found in this way. Notice that three of the operators are marginal[6-8] (i.e., have $x=2$) and hence capable of generating anomalies, including perhaps lines of critical points.

We can see at once one possible reason for identifying the multicritical point with AT-model behavior. The AT model has $x_\sigma=\frac{1}{8}$ and a marginal operator with $x=2$. Our multicritical point has the right index for the order parameter and three[1] marginal operators. We hope that our multicritical point is some point on the AT-model critical line. But where? Here we can make use of the known duality connection[9,10] between the spin formulations of the AT model and the 8V model. To state this connection, assume that both models are critical and that in each model all two-spin interactions are equal, then parametrize the critical lines of the models in terms of their four-spin couplings, called[11] K_4 for the AT model and[6] λ for the 8V model. Specifically use variables $T_4=\tanh 2K_4$ and $t=\tanh 2\lambda$, to describe positions along the critical lines of the two models. Then the duality connection occurs when

$$T_4=-t/(1-t)=T_4(t). \quad (9)$$

When this duality relationship holds then the AT model and the 8V model are identical at their duality point—except for a change in the labeling of the operators. In particular the operator related to the energy density (and hence the specific heat) of the 8V model becomes, after the transformation, the crossover operator of the AT model. If these operators have, respectively, critical indices $x_{\rm CR}{}^{\rm AT}(T_4)$ and $x_\epsilon{}^{\rm 8V}(t)$, then the duality connection[9] implies the statement

$$x_{\rm CR}{}^{\rm AT}[T_4(t)]=x_\epsilon{}^{\rm 8V}(t). \quad (10)$$

At our multicritical point, $x_{\rm CR}=\frac{1}{2}$. From the Baxter[12] solution to the 8V model,

$$x_\epsilon{}^{\rm 8V}(t)=1-(2/\pi)\sin^{-1}t. \quad (11)$$

Hence, our first identification of the multicritical

TABLE I. Our expansion. In columns 1 and 2, O_N, O_N', σ, and σ' stand for $O_N(\vec{r})$, $O_N(r')$, $\sigma(\vec{r})$, and $\sigma(r')$, respectively.

Operator	AT idenditification Symbol	AT idenditification Meaning	x values At (4)	x values Expansion
$\sqrt{2}\,O_{\pm 1/4}$	$\sigma \pm iS - (S\pm i\sigma)$	Order parameter	1/8	$x_\sigma = 1/8$
$(1/2)(O_{1/2}+O_{-1/2})$	$\sigma S = P$	Polarization operator	1/2	$x_P=(1-H)/2$
$(1/i)(O_{1/4}O_{1/4}'-O_{-1/4}O_{-1/4}')$	$\sigma\sigma' - SS'$	Crossover operator	1/2	$x_{\rm CR}=(1+H)/2$
$(1/2)(O_{1/4}O_{-1/4}'+O_{-1/4}O_{1/4}')$	$\sigma\sigma' + SS'$	Energy densities in AT model	2	$x_\epsilon = 2(1-H)$
$(1/2)(O_{1/2}O_{1/2}'+O_{-1/2}O_{-1/2}')$	$\sigma\sigma' SS'$			$x_M=2$
$\tilde{O}_1+\tilde{O}_{-1}$	...			$x_{\tilde{\epsilon}}=2(1+H)$

point of the planar model is

$$t=\sqrt{2}/2;\quad T_4=-(1+\sqrt{2}). \tag{12}$$

If this identification is correct, then essentially trivial calculations based upon applying the Hamiltonian (1) at the special point (4) will define a large number of interesting correlation functions for the AT and 8V models at the identification points (12).

Additional evidence for this identification appears in an expansion[13] along the critical line (5) about the MCP. In this paper we limit ourselves to first-order expansions in $H=2\pi y_0=2\pi y_p$, the results of which are listed in column 5 of the table. (To get a line of fixed points to second order, one must allow a second-order term in $K-4$.) The last three lines in the table are obtained by taking linear combinations of the listed operators to diagonalize the perturbation.

There is now very clear evidence that we are indeed examining the AT model. First, $x_\sigma=\frac{1}{8}$—as proposed by Barber and Baxter.[14] Also, to first order in the expansion parameter

$$x_p^{\,\mathrm{AT}}(T_4)=\tfrac{1}{4}x_\epsilon^{\,\mathrm{AT}}(T_4), \tag{13a}$$

$$x_{\mathrm{CR}}^{\,\mathrm{AT}}(T_4)=[x_\epsilon^{\,\mathrm{AT}}(T_4)]^{-1}. \tag{13b}$$

These statements are consistent with all the data we have about the AT model. In particular, Eq. (13a) was proposed by Enting[15] on the basis of data near $T_4=\frac{1}{2}$ and $T_4=0$ and in analogy to a similar relation[16] which seems to hold for the 8V model. They both hold[15] to first order in T_4 near the decoupling point $T_4=0$ and are consistent with Enting's result[6] for the Potts point, $T_4=\frac{1}{2}$. These relations may be exact.

Now let us follow another tack. Follow Ditzian[17] and Enting[11] and assume that the 8V model and the AT model will fall into exactly the same universality class, without any change in the identification of operators. The first consequence of this universality assumption is that we can also identify our MCP with some point of the 8V model in exactly the same manner as we did for the AT model except that the AT variable S is replaced by the 8V variable μ. We can then identify our expression point as the one at which $x_\epsilon^{\,8\mathrm{V}}(t)=2$, or where $\alpha=-\infty$. We know immediately that our point is then the F-model limit of the 8V model, that is it corresponds to the point

$$t=-1;\quad T_4=\tfrac{1}{2}. \tag{14}$$

The point $T_4=\frac{1}{2}$ is where the AT model reduces to the four-state Potts model.

The expansions now imply that near the second identification point

$$x_p^{\,8\mathrm{V}}(t)=\tfrac{1}{4}x_\epsilon^{\,8\mathrm{V}}(t), \tag{15a}$$

$$x_{\mathrm{CR}}^{\,8\mathrm{V}}(t)=[x_\epsilon^{\,8\mathrm{V}}(t)]^{-1}. \tag{15b}$$

To support this identification notice that Eq. (15a) follows from the work of Baxter and Kelland,[16] that duality implies (15b) as a consequence of (13b), and that they are consistent with first order expansions[15] about $t=0$. (For $t=0$, the 8V model reduces to a pair of decoupled Ising models.) Thus, our correlation functions at the MCP seem to describe both the F model and the four-state Potts model.

If, as suggested by earlier authors,[17,6] there is a universality relation between the two models, then there must be some function $U(T_4)$ such that when $t=U(T_4)$, the 8V model at $t=\tanh 2\lambda$ and the AT model at $T_4=\tanh 2K_4$ have exactly the same critical behavior. Equation (13b) or (15b) describes this universality condition. When combined with Eqs. (9)–(11) they describe the mapping function U as

$$U(T_4)=\sin\{[2/\pi+1/\sin^{-1}(T_4/1-T_4)]^{-1}\}. \tag{16}$$

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