

## Dependence of Four-Body Observables on the Range of Effective Interactions Like Those of the Unitary Pole Approximation

R. Perne and W. Sandhas

*Physikalisches Institut, Universität Bonn, Bonn, West Germany*

(Received 27 July 1977)

A generalized unitary pole approximation concerning the three-body amplitudes in the kernel of four-body integral equations is introduced. We furthermore study the dependence of the  ${}^4\text{He}$  binding energy and of four-body cross sections upon a position-space cut-off parameter in the effective interactions.

Some effort has been made during the last years to study the four-nucleon system by means of integral equations. Essentially all these investigations are based on reducing the original four-body operator identities to manageable effective two-body equations. Such a reduction procedure has been developed<sup>1</sup> and its efficiency was tested in a first numerical calculation by Alt, Grassberger, and Sandhas.<sup>2</sup>

In more recent investigations remarkable improvement has been achieved by using a better suited *two-body* input (Gaussian<sup>3</sup> instead of Yamaguchi form factors, Malfliet-Tjon potential<sup>4,5</sup>). Moreover, as compared to Refs. 1 and 2, a higher number of separable terms in the Hilbert-Schmidt (or Bateman)<sup>6,7</sup> expansion of the effective interactions is usually incorporated.<sup>4,5,7,8</sup>

Instead of these expansions we introduce in the present Letter a generalization of the unitary pole approximation (UPA) which now, however, concerns the *three-body* amplitudes in the kernel of the four-body equations. We furthermore study the dependence of the  ${}^4\text{He}$  binding energy, and of the  $d+d \rightarrow p+t$  and  $p+t \rightarrow p+t$  cross sections upon a position-space *cut-off* parameter.

The first step of the reduction scheme developed in Refs. 1 and 2 consists in replacing the two-body potentials by (a series of) separable terms, a procedure well known from the three-body problem. Here we restrict ourselves to the usual rank-one separable potentials with Yamaguchi and Gaussian form factors  $g_i(p)$ . The parameters in  $g_i(p)$  are adjusted to the following effective-range values  $a_d = 5.46$  fm,  $a_\Phi = -24.56$  fm,

$r_{0,d} = 1.91$  fm,  $r_{0,\Phi} = 2.66$  fm in the deuteron and antibound-state channels  $i = d, \Phi$ . The two-body transition amplitudes then take the well-known separable form (valid also in the two-body UPA)

$$T_i(p, p'; \hat{E}) = g_i(p) \hat{t}_i(\hat{E}) g_i(p'),$$

$$\hat{t}_i(\hat{E}) = \left\{ \lambda_i^{-1} - m \int_0^\infty dp'' p''^2 \frac{g_i^2(p'')}{mE - p''^2} \right\}^{-1}. \quad (1)$$

Herewith the original four-body identities are reduced to effective three-body equations.

Expanding in a second step also the kernel of these equations into a series of separable terms, an effective two-body formulation is achieved<sup>1,2,7,8</sup> as in the genuine three-body case. Conventionally this step is based on separable expansions of the effective "potentials"  $v_{ij}(z) = \langle g_i | G_0(z) | g_j \rangle$  with "form factors"  $G_i(q, z)$  usually chosen as Sturmian functions. I.e., they are defined as eigenfunctions of the "LS-kernel" built up by the potential  $v_{ij}(z)$  and the "free Green's function"  $t_i(z)$ . Fixing here the energy variable  $z$  in an appropriate way (e.g., subsystem pole energy), the corresponding eigenvalue equation takes the form (4), providing  $z$ -independent form factors  $G_{si}(q)$ . This definition is evidently analogous to the one in the two-body UPA. We, therefore, consider our approach as the correct generalization of the two-body UPA idea to the three-body subsystems. The reduction of the numerical effort achieved in this way is, of course, considerable.

With the approximations discussed above our basic effective equations of the four-nucleon problem read

$$T_{L;sr}^{IS}(q, q'; E) = V_{L;sr}^{IS}(q, q'; E) + \sum_n \int_0^\infty V_{L;sn}^{IS}(q, q''; E) \tau_n(E - a^n q''^2) T_{L;nr}^{IS}(q'', q'; E) q''^2 dq'', \quad (2)$$

where

$$a^n = \begin{cases} 2/3m & \text{for } n = (3+1), \\ 1/2m & \text{for } n = (2+2). \end{cases}$$

The index  $L$  denotes the relative partial waves of the clusters,  $I, S$  are the quantum numbers of the total isospin and spin, respectively, while  $s, r, n$ , collectively denote the fragmentations (3+1) or (2+2) and the further quantum numbers of the clusters. The effective potentials in the Lippmann-Schwinger- (LS-) type equation (2) are

$$V_{L;sr}^{IS}(q, q'; E) = \sum_i \Gamma_{L;sr}^{IS} \int_{-1}^{+1} dx P_L(x) G_{si}(q_1) \hat{t}_i(E - Z^2/m) G_{ri}(q_2) \quad (3)$$

with

$$q_1^2 = q^2 + \frac{1}{9} q'^2 + \frac{2}{3} qq'x; \quad q_2^2 = q'^2 + \frac{1}{9} q^2 + \frac{2}{3} qq'x; \quad Z^2 = \frac{3}{4}(q^2 + q'^2) - \frac{1}{2} qq'x$$

in the (3+1) → (3+1) channel. For (2+2) → (3+1) we have

$$q_1^2 = q^2 + \frac{4}{9} q'^2 - \frac{4}{3} qq'x; \quad q_2^2 = q'^2 + \frac{1}{4} q^2 - qq'x; \quad Z^2 = q'^2 + \frac{3}{4} q^2 - qq'x$$

and for (3+1) → (2+2)

$$q_1^2 = q^2 + \frac{1}{4} q'^2 - qq'x; \quad q_2^2 = q'^2 + \frac{1}{4} q^2 - qq'x; \quad Z^2 = q^2 + \frac{3}{4} q'^2 - qq'x.$$

$\Gamma_{L;sr}^{IS}$  are the usual coefficients due to four-body symmetrization and spin-isospin recoupling.

The three-body UPA form factors are defined by

$$G_{si}^k(q) = \eta_k^{-1} \sum_{j=d,\phi} \Lambda_{ij} \int_0^\infty dq' q'^2 v_{ij}^s(q, q') \hat{t}_j(E_{si} - Q^2) G_{sj}^k(q') \quad (4)$$

with

$$Q^2 = \begin{cases} (3/4m)q'^2 & \text{for (3+1),} \\ (1/m)q'^2 & \text{for (2+2),} \end{cases}$$

i.e., by the effective LS equations of the underlying subsystems ( $\Lambda_{ij}$  are the well-known three-body recoupling coefficients). The potentials  $v_{ij}$  have the forms (5a) and (5b) in the (3+1) and (2+2) cases, respectively,

$$v_{ij}^s(q, q') = \int_{-1}^{+1} dx \frac{g_i(\frac{1}{2}q^2 + q'^2 + qq'x)^{1/2} g_j(\frac{1}{2}q'^2 + q^2 + qq'x)^{1/2}}{E_{si} - m^{-1}(q^2 + q'^2 + qq'x)}, \quad (5a)$$

$$v_{ij}^s(q, q') = \frac{g_i(q)g_j(q')}{E_{si} - m^{-1}(q^2 + q'^2)}. \quad (5b)$$

If the system described by Eq. (4) possesses a bound state (e.g., triton, or deuteron + deuteron, etc.),  $E_{si}$  is adjusted to the respective binding energy, and Eq. (4) has a solution with  $\eta_1 = 1$ . In the other cases, where no bound states exist, we get eigenvalues  $\eta_1 \neq 1$ , fixing  $E_{si}$  at the antibound-state energy. We mention that the effective amplitudes corresponding to the UPA expansion of  $v_{ij}^s$  have the form

$$T_{ij}^s(q, q''; \vec{E}) = G_{si}(q) \tau_s(\vec{E}) G_{sj}(q''); \quad \tau_s(\vec{E}) = \{-\eta_s^{-1} - \int_0^\infty G_{si}^2(q') \hat{t}_i(\vec{E} - Q^2) q'^2 dq'\}^{-1}. \quad (6)$$

Here the normalization has been chosen such that the integral term in Eq. (6) is equal to  $-1$  if  $\vec{E} = E_{si}$ . Comparison of Eq. (6) and Eq. (1) makes the analogy of the three-body UPA introduced here to the usual two-body UPA<sup>9</sup> particularly transparent.

Of remarkable physical interest is the relationship between three-body and four-body observables. This has been considered already by Tjon<sup>4</sup> and Sofianos, Fiedelney, and McGurk<sup>10</sup> by varying the two-body input. In the present investigation the same question is studied by changing the *range* of the *effective* potentials. For this purpose expression (5a) is transformed to position space. Introducing cut-off parameters  $a_1, a_2$  for both position variables  $R_1$  and  $R_2$  of the nonlocal potential and retransforming to momentum space we obtain

$$\bar{v}_{ij}^s(q, q') = (4/\pi^2) \int_0^\infty \int_0^\infty f_0(q, q''; a_1) v_{ij}^s(q'', q''') f_0(q''', q'; a_2) q''^2 q'''^2 dq'' dq''', \quad (7)$$

$$f_0(q, q'; a_k) = (a_k/2qq') [j_0((q-q')a_k) - j_0((q+q')a_k)],$$

where  $j_0(qa_k)$  is the spherical Bessel function. Such a procedure allows us to simulate the repulsion in the three-body system. We mention that the various three-body binding energies found by varying  $a_1$

and  $a_2$  independently could also be achieved by varying only  $a_1$  and putting  $a_2 = \infty$ . This simplifying choice is used in the following.

The difficulties originating from the sign oscillations of  $j_0$ , characteristic of entire functions, suggest to solve Eq. (4) with potential (7) by expanding the unknown form factors in an appropriate set, complete in the domain where  $\bar{v}_{ij}(R_1, q')$  differs from zero. We choose the set proposed by Brayshaw,<sup>11</sup> namely the eigenfunctions of the two-particle Lippmann-Schwinger equation in the case of a unit square-well potential with range  $a_1$ . The expansion of the form factors converges quite rapidly (to better than 0.5% after the seventh term).

The simplifying approximation procedure just described has also been applied to the potential (5b) of the (2+2) channel. But since there the cut-off parameter is introduced *only* for the technical reasons explained,  $a_1$  has been chosen so large that its effect on the solution becomes negligible ( $a_1 > 6$  fm). Note that the analytic representation of the form factors  $G(q)$  achieved in this way is particularly useful in scattering calculations, where the contour deformation method requires the knowledge of  $G(q)$  for complex argument.

To obtain the  ${}^4\text{He}$  binding energy we use matrix inversion with the Gaussian net of  $20 \times 20$  mesh points. The angular integration of the effective potentials has been performed with sixteen Gauss points. A numerical accuracy of better than 1% has been estimated by varying the distribution and number of mesh points.

Table I shows the correlation between the triton and  ${}^4\text{He}$  binding energies for several cut-off parameters  $a_1$ . In Fig. 1 we plot  $E_\alpha$  against  $E_t$

TABLE I. Triton binding energies ( $E_t$ ) and  ${}^4\text{He}$  binding energies ( $E_\alpha$ ) as a function of the cut-off parameter  $a_1$ .

Yamaguchi			Gauss		
$a_1$ (fm)	$E_t$ (MeV)	$E_\alpha$ (MeV)	$a_1$ (fm)	$E_t$ (MeV)	$E_\alpha$ (MeV)
2.5	-6.47	-23.38	3.5	-7.72	-26.65
2.7	-7.32	-27.14	4.5	-8.35	-28.73
3.0	-8.33	-31.42	5.0	-8.81	-31.85
3.25	-8.93	-33.81	6.0	-8.93	-33.02
3.5	-9.29	-35.02			
4.5	-9.79	-36.93			
5.0	-10.16	-40.11			
6.0	-10.20	-41.29			
7.0	-10.21	-41.86			

and compare the results with the Tjon line.<sup>4</sup> The strong *linear* relation between  $E_\alpha$  and  $E_t$ , emphasized by Tjon,<sup>4</sup> is found also in our calculation, despite the fact that  $E_t$  is not changed by varying the two-body input but by cutting off the long-range parts of the effective potentials in the (3+1) channel. However, the slopes of the straight lines in the Tjon plot turn out to be different for different two-body potentials. This observation is supported by the fact that the binding energies obtained by Narodetskii<sup>8</sup> and Kharchenko<sup>7</sup> for Yamaguchi potentials and by Becker<sup>3</sup> and Kröger<sup>12</sup> in the Gauss case lie just on the corresponding lines in our calculation. We finally mention that for parameters  $a_1$  which lead to the exact experimental value of  $E_t$  the  ${}^4\text{He}$  binding energy is reasonably well approximated.

In order to get a first crude insight into the influence of the cut-off parameter  $a_1$  on cross sections, we apply the  $K$ -matrix Born approximation introduced in Ref. 2. The results, presented in Fig. 2, show that the variation of  $a_1$  mainly affects the forward direction of the reaction  $d+d \rightarrow t+p$ . As expected, a stronger repulsion, simulated by a smaller  $a_1$ , reduces the magnitude of the forward peak. We particularly emphasize

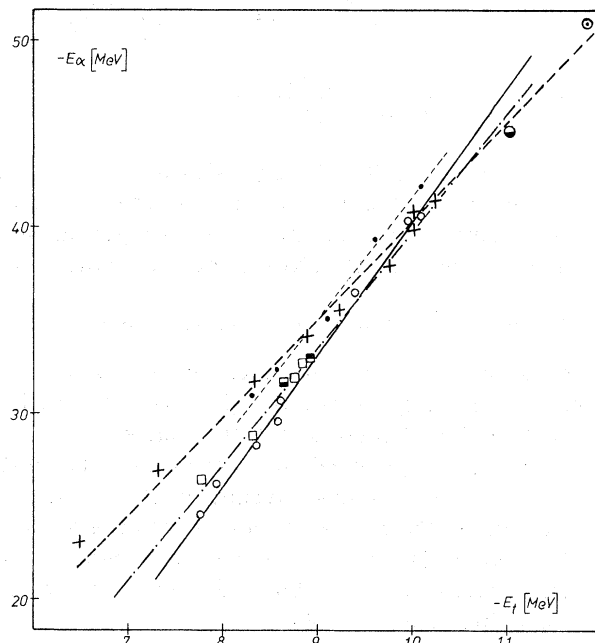


FIG. 1. Tjon plot of  $E_\alpha$  against  $E_t$  for cut-off parameters  $a_1$  tabulated in Table I, using Yamaguchi form factors ---X---X, and Gaussian form factors ---□---. The results are compared with those of Ref. 4, ---○---; Ref. 10, -.-.-□-.-.-; Ref. 8, ·····; Ref. 7, ———; Ref. 3, ———; and Ref. 12, ———.

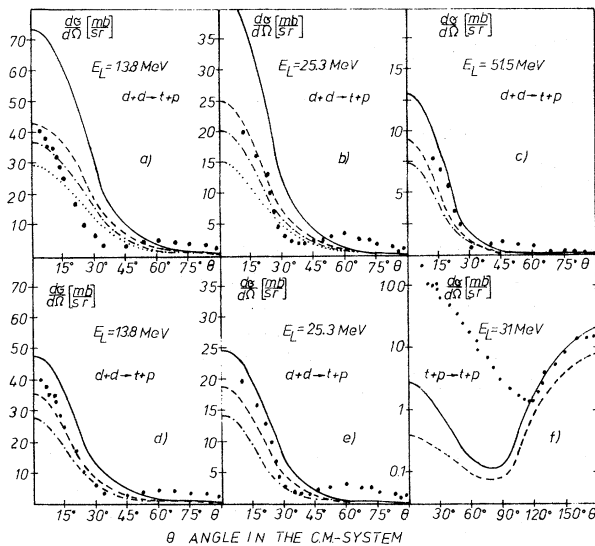


FIG. 2. Differential  $d+d \rightarrow p+t$  and  $p+t \rightarrow p+t$  cross sections. Depicted are (a)–(c) results for Yamaguchi form factors and cut-off parameters  $a_1 = 5$  fm (—),  $a_1 = 3.25$  fm (---),  $a_1 = 3$  fm (— · —),  $a_1 = 2.7$  fm (···). Furthermore, (d)–(f) results for Gaussian form factors and cut-off parameters  $a_1 = 5$  fm (—),  $a_1 = 4.5$  fm (---), and  $a_1 = 3.5$  fm (— · —). Experimental points are from Refs. 13–16.

that the best fit is obtained for a value of the cut-off parameter which just leads to the experimental triton binding energy and consequently to a reasonable result for  $E_\alpha$ . For completeness we have added in Fig. 2 one cross section for the process  $t+p \rightarrow t+p$ , despite the fact that the  $K$ -matrix Born approximation is known to be less justified in this case. Calculations which go beyond the  $K$ -matrix approximation are in progress.

Summarizing we conclude that the three-body UPA proposed here represents a very efficient method for reducing the numerical effort in four-body calculations. Its clear physical meaning, moreover, provides a natural tool for studying the dependence of four-body observables on variations of the effective potentials.

One of us (R.P.) was supported by the Studienstiftung des Deutschen Volkes.

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