Nonlinear Saturation of the Dissipative Trapped-Electron Instability

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It is shown that trapped-electron–induced scattering can be dominant over nonlinear ion Landau damping in the saturation of short-wavelength, dispersive, trapped-electron instabilities in tokamaks. Trapped-electron–induced scattering transfers the wave energy to shorter wavelengths, where it can be dissipated by ion viscosity.

Dissipative trapped-electron instabilities are important in both current and future generations of tokamaks.[1] Although the linear theory of these modes has been studied in considerable detail, most treatments of the nonlinear evolution and saturation of the instabilities, and the associated anomalous transport, have been based on rather primitive models, such as (i) an *ad hoc* relation $\gamma \sim k_\perp^2 D$ to estimate the anomalous diffusion coefficient[1]; (ii) a free-energy argument to estimate the saturation level[2]; or (iii) a one-dimensional resonant mode-coupling process.[3] Approaches (i) and (ii) give upper bounds for the transport coefficients and are, presumably, to be employed only when there are no other mechanisms that give lower saturation levels. Approach (iii) is unsatisfactory for two reasons: First, the nonlinearities due to $\vec{E}\times\vec{B}$ convection are intrinsically two-dimensional; and, second, most of the unstable spectrum lies at short wavelengths[2,3] $[k_\perp^2\rho_i^2(1+T_e/T_i)\sim 1]$, where strong dispersion renders resonant mode coupling ineffective.

In this Letter, we present a new formulation of the nonlinear theory, which includes both two-dimensional nonlinearities and nonresonant mode-coupling interactions.[4] We find that there are two effective, but competing, nonlinear mechanisms, namely trapped-electron–induced scattering and nonlinear ion Landau damping.[4] For typical tokamak parameters, our analysis demonstrates that trapped-electron–induced scattering can dominate over nonlinear ion Landau damping and, in this case, the instabilities are saturated by the spectral transfer of unstable wave energy to the shorter-wavelength (stable) regime, where dissipation due to ion viscosity would occur.

To simplify the analysis, we consider a plane slab geometry, taking $\vec{B}$ in the z direction, and making the usual local WKB approximation for the perturbation fields. Thus, the effects of magnetic shear, toroidal geometry, and radial eigenmode structure are neglected. Although we ignore the destabilizing effect of electron ∇B-drift resonances,[5] our general formulation could easily be extended to include these terms.

The electron dynamics may be described by the drift-kinetic equation with an energy-dependent, number-conserving, Krook model to treat the collisional detrapping processes. Assuming $\omega \sim \nu_{\rm eff} \ll \epsilon^{1/2} v_{Te}/qR$, where q is the safety factor, $\epsilon = r/R$, and $v_{Te} = (2T_c/m_e)^{1/2}$, we may average over the "bounce" motion of the trapped electrons. We obtain a trapped-electron perturbation given by

$$f_e^{\,t}(\vec{k}) = \Phi(\vec{k}) f_e^{(0)} + h_e(\vec{k}), \quad \Phi(\vec{k}) = e\varphi(\vec{k})/T_e, \tag{1}$$

and, in linear order, we have

$$h_c^{(1)}(\vec{k}) = -\frac{\omega_{\vec{k}} - \omega_{*\vec{k}}^{\,T}}{\omega_{\vec{k}} + i\nu_e(v)} \Phi(\vec{k}) f_e^{(0)}. \tag{2}$$

Here, $\nu_e(v) = \nu_{\rm eff}/\bar{v}^3$, $\bar{v} = v/v_{Te}$, $\omega_{*\vec{k}}^{\,T} = \omega_{*\vec{k}}[1 - \eta_e(\frac{3}{2} - \bar{v}^2)]$, $\eta_e = d\ln T_e/d\ln n$, $\omega_{*\vec{k}} = k_y c T_e/eBr_n$, and $r_n^{-1} = |d\ln n/dx|$. In the case being considered, resonant electron effects involving untrapped electrons are smaller than the trapped-electron dissipative effects,[6] so that the untrapped-electron perturbation $f_e^{\,u}(\vec{k})$ is given simply by $f_e^{\,u}(\vec{k}) = \Phi(\vec{k}) f_e^{(0)}$. For the ions, trapping effects are negligible, assuming $\omega \gg \epsilon^{1/2} v_{Ti}/qR$, where $v_{Ti} = (2T_i/m_i)^{1/2}$. The ion dynamics may be described by the drift-kinetic equation generalized to the case $k_\perp\rho_i \sim 1$. We obtain an ion perturbation given by

$$f_i(\vec{k}) = -\tau\Phi(\vec{k}) f_i^{(0)} + h_i(\vec{k}) \exp[ik_\perp\rho\sin(\theta_{\vec{k}} - \alpha_{\vec{k}})] \tag{3}$$

and, in linear order, we have

$$h_i^{(1)}(\vec{k}) = \frac{\tau\omega_{\vec{k}} + \omega_{*\vec{k}}}{\omega_{\vec{k}} - k_\parallel v_\parallel} \Phi(\vec{k}) J_0(k_\perp\rho) f_i^{(0)}, \tag{4}$$

where $\tau = T_e/T_i$, $\rho = v_\perp/\Omega_i$, $\Omega_i = eB/m_i c$, $\sin\theta_{\vec{k}} = v_y/v_\perp$, and $\sin\alpha_{\vec{k}} = k_y/k_\perp$. From the quasineutrality condition $n_e^{\,t} + n_e^{\,u} - n_i = 0$, we obtain the dispersion relation

$$\mathcal{E}(\vec{k}) \equiv 1 + \tau + (\tau + \omega_{*\vec{k}}/\omega_{\vec{k}})\xi_{\vec{k}} Z(\xi_{\vec{k}})\Gamma(b_{\vec{k}}) - (2\epsilon)^{1/2}\langle(\omega_{\vec{k}} - \omega_{*\vec{k}}{}^{T})/[\omega_{\vec{k}} + i\nu_e(v)]\rangle = 0 . \tag{5}$$

Here, the term in angular brackets denotes an average over a Maxwellian distribution, $\Gamma(b_{\vec{k}}) = I_0(b_{\vec{k}}) \times \exp(-b_{\vec{k}})$, $b_{\vec{k}} = \frac{1}{2}k_\perp{}^2\rho_i{}^2$, $\rho_i = v_{Ti}/\Omega_i$, $\xi_{\vec{k}} = \omega_{\vec{k}}/|k_\parallel|v_{Ti}$, and Z is the plasma-dispersion function. In the limit $\xi_{\vec{k}} > 1$, we obtain

$$\omega_{\vec{k}} = \omega_{*\vec{k}}\Gamma(b_{\vec{k}})/H(b_{\vec{k}}), \tag{6}$$

$$\frac{\gamma_{\vec{k}}}{\omega_{\vec{k}}} = \left[(2\epsilon)^{1/2}\,\mathrm{Im}\left\langle\frac{\omega_{\vec{k}} - \omega_{*\vec{k}}{}^{T}}{\omega_{\vec{k}} + i\nu_{\mathrm{eff}}/\bar{v}^{-3}}\right\rangle - \delta_{\vec{k}}{}^{L} - \delta_{\vec{k}}{}^{V}\right][H(b_{\vec{k}})]^{-1}. \tag{7}$$

Here, $H(b_{\vec{k}}) = 1 + \tau - \tau\Gamma(b_{\vec{k}})$, and $\delta_{\vec{k}}{}^{L} = (\pi/2)^{1/2}(\tau + \omega_{*\vec{k}}/\omega_{\vec{k}})\Gamma(b_{\vec{k}})\xi_{\vec{k}}\exp(-\xi_{\vec{k}}{}^2)$, a term describing the ion Landau damping. We have also introduced an ion viscous damping $\delta_{\vec{k}}{}^{V}$; for $\omega_{\vec{k}} \gg \nu_{ii}$, this is given by[7] $\delta_{\vec{k}}{}^{V} = 7\tau(1 + \omega_{*\vec{k}}/\omega_{\vec{k}}\tau)b_{\vec{k}}{}^2\nu_{ii}/10\omega_{\vec{k}}$ for $b_{\vec{k}} < 1$, and $\delta_{\vec{k}}{}^{V} = 3(\pi+1)\tau(1 + \omega_{*\vec{k}}/\omega_{\vec{k}}\tau)b_{\vec{k}}{}^{1/2}\nu_{ii}/8\pi^{1/2}\omega_{\vec{k}}$ for $b_{\vec{k}} > 1$. In this treatment of the ions, we have neglected ion temperature gradients. For $\epsilon \sim \frac{1}{4}$, $\eta_e \sim 1$, and $\tau \sim 1-2$, the maximum growth rate arising from the trapped-electron term is typically[2,3,5] such that $\gamma_{\vec{k}}/\omega_{\vec{k}} \sim 10^{-1}$. Moreover, the modes with large growth rates form a fairly broad spectrum centered around $\tau b_{\vec{k}} \sim 1$ and, for typical parameters of present and future tokamaks, have $\omega_{\vec{k}} > \nu_{\mathrm{eff}}$.

We now consider the nonlinear evolution of the instability. Since the unstable modes are short-wavelength dispersive waves, for which the relevant mode-coupling processes are nonresonant, we must retain terms up to third order in the perturbations. However, $\gamma_{\vec{k}}/\omega_{\vec{k}}$ and ρ_i/r_n may both be treated as small parameters. The nonlinearities arise primarily from nonlinear $\vec{E}\times\vec{B}$ convection terms in the drift-kinetic equations. We may write the quantities $h_{e,i}$ appearing in Eqs. (1) and (3) as perturbation expansions $h_{e,i} = h_{e,i}{}^{(1)} + h_{e,i}{}^{(2)} + h_{e,i}{}^{(3)}$. In second order we have a set of *virtual* modes, with wave vector $\vec{q} = \vec{k} - \vec{k}'$ and frequencies $\omega_{\vec{q}} = \omega_{\vec{k}} - \omega_{\vec{k}'}$, where $\vec{k}$ and $\vec{k}'$ denote wave vectors of the linear normal modes. The perturbed distribution functions for these virtual modes are given, in terms of the second-order virtual-mode potential $\Phi(q)$, by expressions formally identical with those of Eqs. (1)–(4), together with explicitly nonlinear terms:

$$h_i{}^{(2)}(\vec{q}) = \frac{i\Omega_i\rho_s{}^2}{\omega_{\vec{q}} - q_\parallel v_\parallel}\left(\frac{\omega_{*\vec{k}}}{\omega_{\vec{k}}} - \frac{\omega_{*\vec{k}'}}{\omega_{\vec{k}'}}\right)\vec{k}\times\vec{k}'\cdot\vec{e}_z\Phi^*(\vec{k}')\Phi(\vec{k})J_0(\vec{k}_\perp{}'\rho)J_0(\vec{k}_\perp\rho)f_i{}^{(0)}, \tag{8}$$

$$h_e{}^{(2)}(\vec{q}) = \frac{i\Omega_i\rho_s{}^2}{\omega_{\vec{q}} + i\nu_e(v)}\left(\frac{\omega_{*\vec{k}}{}^{T}}{\omega_{\vec{k}}} - \frac{\omega_{\vec{k}'}{}^{T}}{\omega_{\vec{k}'}}\right)\vec{k}\times\vec{k}'\cdot\vec{e}_z\Phi^*(\vec{k}')\Phi(\vec{k})f_c{}^{(0)}, \tag{9}$$

where $\rho_s = v_s/\Omega_i$ with $v_s = (T_e/m_i)^{1/2}$. In the ion term given in Eq. (8), we have assumed $|\omega_{\vec{k}}| \gg |k_\parallel v_\parallel|$ and $|\omega_{\vec{k}'}| \gg |k_\parallel{}' v_\parallel|$. In the electron term given in Eq. (9), noting that $|\omega_{\vec{k}}|, |\omega_{\vec{k}'}| \gg \nu_{\mathrm{eff}}$, we have assumed that $\omega_{\vec{k}}, \omega_{\vec{k}'} > \nu_e(v)$. Also, noting that the nonlinear interactions take place for $|\omega_{\vec{q}}| = |\omega_{\vec{k}} - \omega_{\vec{k}'}| \ll \omega_{\vec{k}}, \omega_{\vec{k}'}$ we have kept ω_q compared with $q_\parallel v_\parallel$ and $\nu_e(v)$. It is a straightforward matter to integrate Eqs. (8) and (9) over velocities, and to substitute into the quasineutrality condition, to obtain the second-order virtual-mode potential $\Phi(q)$. In doing so, we again make use of the fact that the nonlinear interactions take place for $|\omega_{\vec{q}}| \ll \omega_{\vec{k}}, \omega_{\vec{k}}$ and, accordingly, we set $\omega_{\vec{k}} = \omega_{\vec{k}'}$, in the factors in parentheses on the right in Eqs. (8) and (9). We obtain

$$\mathcal{E}(\vec{q})\Phi(\vec{q}) = -(i\Omega_i\rho_s{}^2/\omega_{\vec{k}})\vec{k}\times\vec{k}'\cdot\vec{e}_z\Phi^*(\vec{k}')\Phi(\vec{k})[F_1(\vec{k},\vec{k}')\chi_i(\vec{q}) + \chi_e(\vec{q})], \tag{10}$$

where $F_1(\vec{k},\vec{k}') = \langle J_0(k_\perp\rho)J_0(k_\perp{}'\rho)J_0(q_\perp\rho)\rangle$, $\mathcal{E}(\vec{q}) \simeq \Gamma(b_{\vec{q}})\chi_i(\vec{q}) + \chi_e(\vec{q})$, and

$$\chi_i(\vec{q}) = \frac{\omega_{*\vec{q}}}{\omega_{\vec{q}}}\xi_{\vec{q}}Z(\xi_{\vec{q}}); \quad \chi_e(\vec{q}) = (2\epsilon)^{1/2}\left\langle\frac{\omega_{*\vec{q}}{}^{T}}{\omega_{\vec{q}} + i\nu_e(v)}\right\rangle. \tag{11}$$

The remainder of the calculation is fairly straightforward and follows standard procedures.[4] We have third-order perturbations

$$h_i{}^{(3)}(\vec{k}) = -\frac{i\Omega_i\rho_s{}^2}{\omega_{\vec{k}} - k_\parallel v_\parallel}\sum_{\vec{k}'}\vec{k}\times\vec{k}'\cdot\vec{e}_z\{\Phi(\vec{k}')J_0(k_\perp{}'\rho)[h_i{}^{(1)}(\vec{q}) + h_i{}^{(2)}(\vec{q})] - \Phi(\vec{q})J_0(q_\perp\rho)h_i{}^{(1)}(\vec{k})\}, \tag{12}$$

$$h_e{}^{(3)}(\vec{k}) = -\frac{i\Omega_i\rho_s{}^2}{\omega_{\vec{k}} + i\nu_e(v)}\sum_{\vec{k}'}\vec{k}\times\vec{k}'\cdot\vec{e}_z\{\Phi(\vec{k}')[h_e{}^{(1)}(\vec{q}) + h_e{}^{(2)}(\vec{q})] - \Phi(\vec{q})h_e{}^{(1)}(\vec{k}')\}, \tag{13}$$

where, as before, $\vec{q}=\vec{k}-\vec{k}'$ denotes the wave vector of the second-order virtual mode. Using the quasineutrality condition to third order, and the random-phase approximation, we obtain the wave kinetic equation

$$\frac{1}{2}\frac{\partial N_{\vec{k}}}{\partial t}=\gamma_{\vec{k}}N_{\vec{k}}+\sum_{\vec{k}'}\beta(\vec{k},\vec{k}')N_{\vec{k}}N_{\vec{k}'}. \tag{14}$$

Here, we have introduced the number of drift-wave plasmons, given by $N_{\vec{k}}=[\partial\mathcal{E}(\vec{k})/\partial\omega_{\vec{k}}]|\Phi(\vec{k})|^2$, where $\partial\mathcal{E}(\vec{k})/\partial\omega_{\vec{k}}=H(b_{\vec{k}})/\omega_{\vec{k}}$. The coupling coefficients are given by

$$\beta(\vec{k},\vec{k}')=-\frac{\tau^2\Omega_i^{\,2}b_{\vec{k}}b_{\vec{k}'}\sin^2(\alpha_{\vec{k}}-\alpha_{\vec{k}'})}{H(b_{\vec{k}})H(b_{\vec{k}'})}\left(F(\vec{k},\vec{k}')\,\mathrm{Im}\frac{\chi_i(\vec{q})\chi_e{}^t(\vec{q})}{\mathcal{E}(\vec{q})}+G(\vec{k},\vec{k}')\,\mathrm{Im}\frac{\chi_i(\vec{q})^2}{\mathcal{E}(\vec{q})}\right), \tag{15}$$

where

$$F(\vec{k},\vec{k}')=\Gamma(b_{\vec{q}})-2\langle J_0(k_\perp\rho)J_0(k_\perp{}'\rho)J_0(q_\perp\rho)\rangle+\langle J_0{}^2(k_\perp\rho)J_0{}^2(k_\perp{}'\rho)\rangle, \tag{16}$$

$$G(\vec{k},\vec{k}')=\Gamma(b_{\vec{q}})\langle J_0{}^2(k_\perp\rho)J_0{}^2(k_\perp{}'\rho)\rangle-\langle J_0(k_\perp\rho)J_0(k_\perp{}'\rho)J_0(q_\perp\rho)\rangle^2. \tag{17}$$

It should be noted that, although the quantities $\mathrm{Im}\chi_e{}^t(\vec{k})$ and $\mathrm{Im}\chi_e{}^t(\vec{k}')$ are of order $\gamma_{\vec{k}}/\omega_{\vec{k}}$ and $\gamma_{\vec{k}'}/\omega_{\vec{k}}$, the quantity $\mathrm{Im}\chi_e{}^t(\vec{q})$ can be of order unity, since the wave vector $\vec{q}$ corresponds to a virtual mode, rather than a normal mode. More specifically, due to the energy dependence of the collision frequency, $\nu_e(v)=\nu_{\mathrm{eff}}/\bar{v}^3$, only low-energy ($\bar{v}<1$) electrons contribute to the integrals in $\mathrm{Im}\chi_e{}^t(\vec{k})$ and $\mathrm{Im}\chi_e{}^t(\vec{k}')$, whereas with $|\omega_{\vec{q}}|\ll|\omega_{\vec{k}}|, |\omega_{\vec{k}'}|$, electrons up to high energies ($\bar{v}\gg 1$) (i.e., almost the entire population of electrons) contribute to the integral in $\mathrm{Im}\chi_e{}^t(\vec{q})$. Thus, we have $|\mathrm{Im}\chi_e{}^t(\vec{k})|, |\mathrm{Im}\chi_e{}^t(\vec{k}')|\ll \mathrm{Im}\chi_e{}^t(\vec{q})|$, and the weak turbulence expansion is valid. As functions of $\omega_{\vec{q}}$, the functions $\mathrm{Im}\chi_i{}^t(\vec{q})$ and $\mathrm{Im}\chi_e{}^t(\vec{q})$ are sharply peaked around $\omega_{\vec{q}}=0$. For $|\omega_{\vec{k}}|, |\omega_{\vec{k}'}|>\nu_{\mathrm{eff}}, |k_\| v_i|, |k_\|{}'v_e|$ we can approximate the quantities in Eq. (15) by δ functions:

$$\mathrm{Im}[\chi_i(\vec{q})\chi_e{}^t(\vec{q})/\mathcal{E}(\vec{q})]=-\pi\delta(\omega_{\vec{k}}-\omega_{\vec{k}'})(\omega_{*\vec{k}}-\omega_{*\vec{k}'})(2\epsilon)^{1/2}/[\Gamma(b_{\vec{q}})-(2\epsilon)^{1/2}], \tag{18}$$

$$\mathrm{Im}[\chi_i(\vec{q})^2/\mathcal{E}(\vec{q})]=\pi\delta(\omega_{\vec{k}}-\omega_{\vec{k}'})(\omega_{*\vec{k}}-\omega_{*\vec{k}'})/[\Gamma(b_{\vec{q}})-(2\epsilon)^{1/2}]. \tag{19}$$

Equations (18) and (19) have been derived by using the appropriate asymptotic forms of the functions $\chi_e{}^t(\vec{q})$, $\chi_i(\vec{q})$, and $\mathcal{E}(\vec{q})$.

In Eq. (15), the term in $G(\vec{k},\vec{k}')$ corresponds to ion-induced scattering[4] (nonlinear ion Landau damping); the term in $F(\vec{k},\vec{k}')$ is a new term, and corresponds to trapped-electron–induced scattering. Since $\beta(\vec{k},\vec{k}')=-\beta(\vec{k}',\vec{k})$, the total number of plasmons is conserved in both scattering processes. Both scattering processes are intrinsically two dimensional. If the wave-vectors $\vec{k}$ and $\vec{k}'$ are in the same direction, then $\sin(\alpha_{\vec{k}}-\alpha_{\vec{k}'})=0$, with the result that $\beta(\vec{k},\vec{k}')=0$. We note also that $F(\vec{k},\vec{k}')\gg 0$ and $G(\vec{k},\vec{k}')>0$. We see that the ion-induced scattering contributes positively to $\beta(\vec{k},\vec{k}')$ if $k_y<k_y{}'$; i.e., wave energy is transferred from short wavelengths to long wavelengths. On the other hand, the trapped-electron–induced scattering contributes negatively to $\beta(\vec{k},\vec{k}')$ if $k_y<k_y{}'$; i.e., wave energy is transferred from long wavelengths to short wavelengths. The relative importance of these two competing processes depends on the relative magnitude of $F(\vec{k},\vec{k}')$ and $G(\vec{k},\vec{k}')$. While the expressions are generally complicated, we can analyze them in two limits. In the limit $b_{\vec{k}}, b_{\vec{k}'}, b_{\vec{q}}<1$, we can show that $F(\vec{k},\vec{k}')\simeq 2b_{\vec{k}}b_{\vec{k}'}\cos^2(\alpha_{\vec{k}}-\alpha_{\vec{k}'})$ and $G(\vec{k},\vec{k}')\simeq b_{\vec{k}}b_{\vec{k}'}\times\cos^2(\alpha_{\vec{k}}-\alpha_{\vec{k}'})$. In the opposite limit $b_{\vec{k}}, b_{\vec{k}'}, b_{\vec{q}}>1$, we find $G(\vec{k},\vec{k}')\leq\Gamma(b_{\vec{q}})\Gamma(b_{\vec{k}})\Gamma(b_{\vec{k}'})<F(\vec{k},\vec{k}')\simeq\Gamma(b_{\vec{q}})$. Thus, trapped-electron–induced scattering dominates in both limits whenever $\epsilon>\frac{1}{8}$, and we expect that for typical ϵ values (e.g., $\epsilon\simeq\frac{1}{4}$) trapped-electron–induced scattering always dominates over nonlinear ion Landau damping. Whenever trapped-electron–induced scattering dominates, the virtual mode must have negative dissipation. On the other hand, our approximation is only valid if $\Gamma_{\vec{q}}>(2\epsilon)^{1/2}$; otherwise, the virtual mode could resonate with an unstable normal mode at very low frequency.[8]

Certain general conclusions can be drawn from Eq. (15). To simplify this discussion, we may restrict ourselves to the case $\tau\gg 1$ and $b_{\vec{k}}\ll 1$, such that $\tau b_{\vec{k}}\sim 1$. In this limit, the small-b expansions of $F(\vec{k},\vec{k}')$ and $G(\vec{k},\vec{k}')$, given above, are valid; moreover $\Gamma_{\vec{q}}\simeq 1$, so that the denominators in Eqs. (18) and (19) remain positive. The dispersion relation is $\omega_{\vec{k}}\simeq\omega_{*\vec{k}}/(1+\tau b_{\vec{k}})$ and the modes with largest frequencies, and largest growth rates,[2,3] have $\tau b_{\vec{k}}^{+} \simeq 1$. With trapped-electron–induced scattering dominating over nonlinear ion Landau damping, the unstable wave energy is nonlinearly transferred to modes with higher k_y values, i.e., to shorter-wavelength modes. Since there exist stable modes at short wavelengths due to ion viscous damping, they provide the necessary energy sink to achieve the saturation.

A rough estimate for the saturation amplitudes of the unstable modes can be obtained by balancing the growth rate with trapped-electron–induced scattering, i.e., by setting $N_{\vec{k}}^{+} \sim \gamma_{\vec{k}}^{+}/\beta(\vec{k}, \vec{k}')$. Assuming $\epsilon \sim \frac{1}{4}$, $\tau b_{\vec{k}}^{+} \sim 1$, and $\omega_{\vec{k}}^{+} \sim \omega_{*\vec{k}}^{+}/2$, we estimate that $\beta(\vec{k}, \vec{k}') \sim (\pi/20)\Omega_i^2/\tau^2$, giving $|\tilde{n}_e/n_0| \simeq |\Phi_{\vec{k}}^{+}| \sim (\omega_{\vec{k}}^{+} N_{\vec{k}}^{+}/2)^{1/2} \sim (\tau\rho_s/r_n)(\gamma_{\vec{k}}^{+}/\omega_{\vec{k}}^{+})^{1/2} \sim 1\%$ typically. In the quasilinear approximation, we can estimate the cross-field particle diffusion coefficient as $D \simeq |k_y \varphi_{\vec{k}}^{+}/B|^2 \gamma_{\vec{k}}^{+}/\omega_{\vec{k}}^{+2} \sim (2\tau^2\rho_s/r_n)(\gamma_{\vec{k}}^{+}/\omega_{\vec{k}}^{+})^2 D$, where $D_B = cT_e/cB$. For modes with similar wavelengths (i.e., $k_y\rho_s \sim k_\perp\rho_s \sim 1$) conventional estimates based on $D \sim \gamma_k/k_\perp^2$ would give $D \sim (\rho_s/r_n)(\gamma_{\vec{k}}^{+}/\omega_{\vec{k}}^{+})D_B$. Thus, for $\gamma_{\vec{k}}^{+}/\omega_{\vec{k}}^{+} \sim 10^{-1}$ (as is typical of these modes) and $\tau \sim 2$, our new estimate gives a value for D comparable to the conventional one. On the other hand, the largest contributions to $\gamma_k/k_\perp^2$ arise from somewhat smaller $k_\perp$ values [typically,[5] $k_\perp^2\rho_s^2 \sim 0.2$, giving $D \sim 0.2(\rho_s/r_n)D_B$], for which resonant three-wave coupling also needs to be considered.

To obtain the detailed nonlinear dynamics and the saturated spectrum of modes, a computational solution of Eq. (14) is needed. Resonant three-wave coupling can also be included in such a treatment.

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High-β Tokamaks

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A model allowing the calculation of very high-β tokamak equilibria with choice of β, β_p, and q_0 is described. One of these equilibria having $\beta = 12\%$ has been shown to be stable to low-n internal modes.

It has been widely believed that, with the constraint $q \gtrsim 1$, there is a limit to the achievable β in tokamaks imposed by equilibrium limitations. Clarke[1] has drawn attention to the argument of Mukhovatov and Shafranov[2] showing that this is not the case. Because of the crucial importance of obtaining a high value of β for an economic fusion reactor, it is now essential to understand the stability properties of high-β configurations. We have used a method of calculating high-β equilibria with a chosen value of q on axis (q_0) which is reminiscent of the observed behavior of tokamaks and which leads to stable configurations with higher values of β. Stability against low-n internal modes, including the effects of ballooning, is obtained for $\beta = 12\%$ (where $\beta = 2\int p\,d\tau/\int B^2\,d\tau$).

This value of β represents a considerable improvement over the average β of 3% obtained in our previous calculation using a simple equilibrium.[3] Since the completion of the calculations described here, two other papers dealing with the