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Topological Excitations in U(1)-Invariant Theories

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A class of U(1)-invariant theories in d dimensions is introduced on a lattice. These theories are labeled by a simplex number s, with $1 \leq s < d$. The case with $s=1$ is the X-Y model; and $s=2$ gives compact photodynamics. An exact duality transformation is applied to show that the U(1)-invariant theory in d dimensions with simplex number s is the same as a similar theory in d dimensions but which is Z_∞-invariant and has simplex number $\tilde{s}=d-s$. This dual theory describes the topological excitations of the original theory. These excitations are of dimension $\tilde{s}-1$.

One of the most interesting consequences of a theory with a compact symmetry is the possibility of excitations resulting from the periodic nature of the symmetry group. Examples of these topological excitations are pseudoparticles in non-Abelian gauge theories, vortices in the classical X-Y model (thought to describe vortex formation in superfluids), and the spin-alignment interface of, say, the three-dimensional Ising model. The nature of these excitations, in particular their dimensionality, is thought to depend only on very general considerations, such as the space-time dimensionality of the theory and the nature of the symmetry group. For systems with a global symmetry and a local order parameter, Toulouse and Kléman[1] have presented homotopy arguments leading to an expression for the dimensionality of these excitations. However, these general arguments do not provide one with a detailed description of the interactions of these excitations. Moreover, when the theory has more than just a global invariance, these arguments in their simple form are no longer valid.

In this Letter I examine a class of U(1)-invariant theories in arbitrary dimension, d. Each theory in d dimensions is parametrized by an index, s (with $1 \leq s < d$), which defines the nature of the interaction.[2] As s increases, the U(1) symmetry becomes increasingly local. These theories can be transformed by a duality transformation into an equivalent set of theories, also in d dimensions, whose fields are integer valued. This dual formulation can be used to construct the partition functions (simplest at low temperatures) which describe the interactions of the topological excitations of the original, U(1)-invariant theories. The dimensionality, l, of the excitations (points, lines, sheets, etc.) is given by $l=d-s-1$.

The theories can be defined on a hypercubical lattice of dimension d. (The connection with continuum theories is described below.) A simplex of dimension s is an s-dimensional element of the lattice. A vertex of the lattice is a simplex of dimension zero, a link joining two vertices is a simplex of dimension one, an elementary face (defined by four vertices) is a simplex of dimensions two, and so on. The theory with simplex number s has a spin,

$$U_{\alpha_1,\ldots,\alpha_{s-1};j} = \exp i\theta_{\alpha_1,\ldots,\alpha_{s-1};j}, \tag{1}$$

associated with each simplex of dimension $s-1$. The α_i's define the orientation of the simplex, and j refers to a position in the lattice (lattice site). We take the θ's to be antisymmetric under the interchange of any pair of its indices. The interaction is defined by coupling together the $2s$ simplices of dimension $s-1$ which border a simplex of dimension s according to the interaction

$$I_{\{\alpha\}}(j)=\exp\left[\frac{i}{(d-s)!(s-1)!}(\epsilon_{\alpha_1,\ldots,\alpha_s,\beta_1,\ldots,\beta_{d-s}}\epsilon_{\beta_1,\ldots,\beta_{d-s},\nu,\gamma_1,\ldots,\gamma_{s-1}}\Delta_\nu\theta_{\gamma_1,\ldots,\gamma_{s-1};j})\right], \tag{2}$$

where

$$\Delta_\nu\theta_{\gamma_1,\ldots,\gamma_{s-1};j}\equiv\theta_{\gamma_1,\ldots,\gamma_{s-1};j}-\theta_{\gamma_1,\ldots,\gamma_{s-1};j-\hat{\nu}},$$

ϵ is the totally antisymmetric symbol in d dimensions, and a sum over the β's, γ's, and ν is implied.

The partition function is then

$$Z_{d,s}=\int\delta\theta\,\exp\left[\frac{\beta}{2(s)!}\sum(I_{\{\alpha\}}+I_{\{\alpha\}}{}^{+})\right]. \tag{3}$$

The integration is over the set of θ's, $-\pi<\theta\leq\pi$, and the sum in the exponent runs over all the simplices of dimension s, defined by an ordered set $\{\alpha_1,\ldots,\alpha_s\}$ contained in the lattice of dimension d. Using (2), we see that the exponent in (3) is invariant under a uniform rotation of the set of angles which border on a given simplex of dimension $s-2$, so as s increases fewer and fewer angles are involved and the symmetry becomes increasingly local (or, more properly, increasingly directional). The case with $s=1$, which corresponds to the X-Y model, involves a true global symmetry.

Using (2) in (3), we see that the terms in the exponent can be written as cosines of angular differences. For each such term I introduce into (3) the representation

$$e^{\beta\cos\omega}=\sum_n I_n(\beta)e^{in\omega}, \tag{4}$$

so that there is one index, n, for each s-dimensional simplex. After integrating over the set of angles, ω, and dropping some overall constants, (3) can be written as

$$Z_{d,s}=\sum_{\{n\}}\prod_x I_n(\beta)\prod_y\delta(\epsilon_{\alpha_1,\ldots,\alpha_{s-1},\nu,\beta_1,\ldots,\beta_{d-s}}\epsilon_{\beta_1,\ldots,\beta_{d-s},\gamma_1,\ldots,\gamma_s}\Delta_\nu n_{\gamma_1,\ldots,\gamma_s;j}). \tag{5}$$

The product over x is a product over all s-dimensional simplices, and the product over y is a product over all $(s-1)$-dimensional simplices. The Kronecker δ functions arise as a result of the integrations over the θ's, and tie together the indices belonging to all the s-dimensional simplices which share a common $(s-1)$-dimensional border.

The δ-function constraints will be automatically satisfied if I write the n's in the form

$$n_{\gamma_1,\ldots,\gamma_s;j}=\frac{1}{(d-s-1)!}\epsilon_{\gamma_1,\ldots,\gamma_s,\nu,\lambda_1,\ldots,\lambda_{d-s-1}}\Delta_\nu T_{\lambda_1,\ldots,\lambda_{d-s-1};j}. \tag{6}$$

[Note that the n's and the T's can be taken to be antisymmetric under interchange of any pair of indices. This follows from (6) and the antisymmetry of the θ's.] Inverting Eq. (4), I have an integral representation for $I_n(\beta)$, valid for integer n. By writing $I_n(\beta)=\exp[\ln I_n(\beta)]$, the integral representation can be used to expand the exponent in powers of n. Neglecting unessential overall constants, (5) can then be written in the form

$$Z_{d,s}=\sum_{\{n\}}\exp\left[\sum\sum_{p=1}^{\infty}(-1)^p C_p(\beta)\left(\frac{1}{(d-s-1)!}\epsilon_{\gamma_1,\ldots,\gamma_s,\nu,\lambda_1,\ldots,\lambda_{d-s-1}}\Delta_\nu T_{\lambda_1,\ldots,\lambda_{d-s-1};j}\right)^{2p}\right]. \tag{7}$$

The sum in the exponent runs over all the s-dimensional simplices, i.e., ordered sets of $\{\gamma\}$, on the lattice and includes a normalization factor of $1/s!$. Comparison with expressions (2) and (3) shows that (7) has the structure of a partition function for a system in d dimensions with simplex numbers $\tilde{s}=d-s$, but now the angles, T, have discrete, instead of compact, support. In addition, a careful analysis of the steps leading to (7) reveals that the simplices of dimension $\tilde{s}$ are actually simplices on the dual lattice which is obtained by translating the original lattice by half a lattice spacing in every direction. These simplices are in a one-to-one correspondence with the dimension-s simplices of the original lattice.

The $C_p(\beta)$ in (7) can easily be deduced from the inverse of (4) and are essentially cumulants of integrals of powers of φ with the weight $e^{\beta\cos\varphi}$. For large β, it is easy to estimate the $C_p(\beta)$ and I find that $C_1(\beta)\simeq 1/\beta$. This means that in the partition function we can expect substantial contributions from n's that satisfy $n^2 \lesssim \beta$. For large β, $C_2(\beta)\sim 1/\beta^3$, so even though $(n^2)^2\sim\beta^2$ this term will be less important by one power of β than the quadratic term. Higher-order terms will also be less important, and so for large β it is a good approximation to keep only the $p=1$ term. I shall do this in what follows, but it is important to note that none of my symmetry arguments depends on this truncation since all terms in (7) have the same form.

Now, consider what the sum over $\{n\}$ in (7) means. I am to select a set of T's which, through Eq. (6), define a set of n's. I then calculate the contribution of that set of n's to the partition function. Moving on to another set of T's, I repeat the process. But the sum in (7) [or (5)] is over different sets of n's, and so one must be careful not to include two sets of T's which give the some set of n's. This is of course the usual problem of gauge invariance in a functional integral and appears here because the theory defined by (7) has, for $d-s>1$, a discrete local symmetry. In fact, the case with $d-s=2$ corresponds exactly to the gauge symmetry of free photons except that the gauge fields, T_λ, are restricted to take on only integral values.

One can rewrite (7) in a more illuminating form by using the identity

$$\sum_{m=-\infty}^{\infty}\delta(z-m)=\sum_{k=-\infty}^{\infty}e^{i2\pi kz}, \tag{8}$$

where the δ functions here are the Dirac δ functions. Keeping only the $p=1$ term in (7) and using (8), one has

$$Z_{d,s}=\sum_{\{K\}}\int\delta n_{\gamma_1,\ldots,\gamma_s;j}\exp\Big(\sum-\frac{1}{\beta}(\epsilon\Delta T)^2+i2\pi K_{\gamma_1,\ldots,\gamma_s}(\epsilon\Delta T)_{\gamma_1,\ldots,\gamma_s}\Big), \tag{9}$$

where I have suppressed the indices in $\epsilon\Delta T$ [see Eq. (6)]. Here the sum over $\{K\}$ is a sum over a set of integers which are labeled by an unordered set of indices, $\{\gamma\}$. To avoid overall infinite factors, the sum over the K's can be restricted by a gauge condition so that sets of K's which differ from each other only by a total derivative are not included in the sum. In addition, K's whose labels differ from each other only by permutations are not independent, and can be taken to be antisymmetric under the interchange of any two indices.

Finally, I want to rearrange the last term, $K\circ\epsilon\Delta T$ by collecting together coefficients of a given T. This is easily done and the last term becomes

$$\begin{aligned}\sum i2\pi K\cdot(\epsilon\Delta T)&=\sum -i2\pi(\epsilon_{\gamma_1,\ldots,\gamma_s,\nu,\lambda_1,\ldots,\lambda_{d-s-1}}\Delta_\nu K_{\gamma_1,\ldots,\gamma_s;j})T_{\lambda_1,\ldots,\lambda_{d-s-1};j}\\ &\equiv\sum -i2\pi J_{\lambda_1,\ldots,\lambda_{d-s-1};j}T_{\lambda_1,\ldots,\lambda_{d-s-1};j}.\end{aligned} \tag{10}$$

We can now trade in the sum over the set of integers $\{K\}$ for a sum over a set of integers $\{J\}$ defined through (10). Now the J's are not all independent, but must satisfy the representation implied by (10). It is easily seen that this means that

$$\Delta_{\lambda_i}J_{\lambda_1,\ldots,\lambda_{d-s-i};j}=0, \tag{11}$$

where λ_i is any one of the set of indices $\{\lambda\}$. The partition function can therefore be written

$$Z_{d,s}=\sum_{\{J\}}{}'\int\delta n_{\gamma;j}\exp[\sum(-1/\beta)(\epsilon_{\gamma,\nu,\lambda}\Delta_\nu T_{\lambda;j})^2-2\pi iJ_{\lambda;j}T_{\lambda;j}], \tag{12}$$

when γ and λ stand, respectively, for the sets $\{\gamma_1,\ldots,\gamma_s\}$ and $\{\lambda_1,\ldots,\lambda_{d-s-1}\}$, and the prime denotes a restriction to integers defined through (10), and therefore satisfying (11). We can now perform the functional integral over the $n_{\gamma;j}$, and we will end up with a partition function which has the form

$$Z_{d,s}=Z_{d,s}^{(0)}(\beta)\sum_{\{J\}}{}'\exp[\beta\sum_{j,j'}J_{\lambda;j}D_{\lambda,\lambda'}(j,j')J_{\lambda';j}], \tag{13}$$

where $Z^{(0)}$ is a factor independent of J. The precise form of the interaction $D_{\lambda,\lambda'}(j,j')$ depends on s and d, but I can read off several general features from (13). First, recall that the J's are integers and satisfy (12). Therefore, unless $\bar{s}\equiv d-s=1$, I cannot have isolated point excitations. In fact, it is easy to see from (11) that the dimensionality of allowed excitations is $\tilde{s}-1$. So, for instance, $\tilde{s}=2$ allows lines which extend to infinity, or close on themselves (smoke rings), and $\tilde{s}=3$ allows infinite sheets or closed two-dimensional manifolds, etc. These conclusions are consistent with the models which have been studied previously. In particular, it is known that the two- and three-dimensional X-Y models have point and line singularities, respectively,[1,3] and that the compact Abelian gauge theory has point singularities when $d=3$, and line singularities when $d=4$.[4] Of course, in the general case some of these configurations may be ruled out by energy considerations, but that depends more particularly on the form of D.

Next, since (13) is a good approximation to Z at low temperatures, these topological excitations will in general be the relevant ones in this regime, aside from nonsingular excitations such as spin waves (or their generalizations). The partition function of the spin-wave–like excitations is the factor $Z^{(0)}$ in (13). Since one expects in general that a chemical-potential term for the allowed excitations will appear in (13), the spin-wave–like oscillations could well dominate at temperatures below those necessary to excite the J's. These excitations just represent the effect of decompactifying the symmetry. This can be seen by noting that if all the J's are zero, then all K's in (9) can be set equal to zero, and I would then have the theory which would have resulted had I kept only the quadratic forms in the θ's and let $-\infty<\theta<\infty$. Another way to say this is that with all the $J=0$, the T_λ's would be allowed to vary continuously from $-\infty$ to ∞. For example, if $\tilde{s}=2$ I would just have free photons. Whether in a given system there actually exists a noncompact phase could depend on the details of D, but it is a scenario allowed by the form (13).[5]

There are two more comments to make about my result. First, the result (13) is properly gauge invariant. The reason is that although the specific form of D may depend on how I defined the functional integral in (12), the J's are restricted to satisfy the correct condition to guarantee that any allowed configuration of J's will give a gauge-invariant contribution to Z. For instance, for $\tilde{s}=2$, the J's satisfy $\Delta_\mu J_\mu=0$. Second, it is important to stress that even though I have formulated this approach on a lattice, there is a well-defined connection with continuum theories. One may worry that because there is no spatial continuity on a lattice, it is not possible to define topological singularities. Strictly speaking, this is true. However, a careful study of the $d=2$ X-Y model ($s=1$) reveals that the low-temperature partition functions of the lattice and of an analogous continuum theory are identical, and that the integer-valued excitations of the lattice become in a well-defined way the vortices of the continuum.[6]

More specific features of the theories described here will be discussed elsewhere, as will the extension of our method to non-Abelian symmetries.

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[1]G. Toulouse and M. Kléman, J. Phys. (Paris), Lett. 37, L149 (1976).

[2]A similar hierarchy of theories has been discussed by F. Wegner, J. Math. Phys. (N.Y.) 12, 2259 (1971), for the Ising-like case in which the symmetry group is Z_2.

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[4]A. M. Polyakov, Phys. Lett. 59B, 82 (1975); M. Peskin, private communication.

[5]Such a situation is in fact realized in the $d=2$ X-Y model ($s=1$). For details see Ref. 3.

[6]Savit, Ref. 3.