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Scale-Covariant Theory of Gravitation and Astrophysical Applications

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We present generalized Einstein equations, invariant under scale transformations, and study several astrophysical tests. It is assumed that the dynamics of atoms or clocks used as measuring apparatus is given *a priori*. Connection with gauge fields and broken symmetries is made through the cosmological constant.

A physical theory can be considered complete only when it provides the system of units to be used. Examples of complete but unrelated theories are quantum electrodynamics (QED) and gravitation, which define two separate systems of units: one purely atomic, i.e., made of e, m, and h, and the other purely gravitational, made of G, M, and R, where M and R are masses and radii of astronomical bodies.

If the two theories could be unified into one single scheme, then a new system of units would emerge, which need not coincide with either of the two previous ones. Insofar as such a theory does not yet exist, it is logically consistent to consider that today's ratio of electromagnetic to gravitational units has not been the same throughout the evolution of the Universe. We shall therefore consider that e^2/Gm_em_p, which today amounts to 2×10^{39}, can be in general written as $\beta^{-1}(x)$. The normal practice so far has been that of postulating that $\beta(x)$ is independent of space and time, i.e., 10^{40} has been so during the entire evolution of the Universe, a built-in disparity that defies by fiat any attempts to unify the various types of interactions. However, the recent success of gauge fields with broken symmetries[1] can be taken to support the position that any *a priori* assumption about the strength of different forces is unjustifiable. In the spirit of gauge fields with phase transitions,[2] we shall consider that the disparity in strength between atomic and gravitational forces is not a fundamental or intrinsic property of the physical world, but rather is the result of one making measurements when the Universe is 20×10^9 years old and permeated by a 3-K temperature.[3] Within the framework of gauge fields, the high-temperature, high-density scenario that prevailed in the early Universe assured the equality of e^2/hc, G_F, $g^2/4\pi$, etc. The symmetry was broken by the expansion of the Universe, that cooled the temperature below 10^{15} K at a time $t=10^{-10}$ sec. Since that moment till today, the fundamental interactions have looked "different" from one another both in strength as well as range. In this Letter we will consider a somewhat related concept but we shall concentrate exclusively on gravitation. We shall not postulate a phase transition, but rather a smooth change taking place throughout the entire history of the Universe. The guiding principle is, however, the same: In the early phases of the Universe, gravity was as strong (or as weak) and of the same range as any other type of interaction. However, since any variation of G is strictly forbidden within the framework of Einstein equations, in order to implement the idea, we shall resort to two different sets of units, since, as we said, there is no *a priori* reason for electromagnetic and gravitational units to be constant multiples of each other. Following Dirac,[4] we shall assume that *atomic units* are such that e, h, and m are constant with respect to atomic time, and that *Einstein units* are defined so that G, M, and R are constant with respect to gravitational time. Einstein equations with G constant are understood to be valid *only* in Einstein units, i.e., with respect to clocks set up by astronomical systems whose dynamics are solely determined by gravity. However, G can vary with respect to atomic units as much as e, h, and m can vary with respect to Einstein units. In practice, since our measuring apparatus are made of atoms, Einstein units are of limited use. If we call $d\bar{s}^2 = \bar{g}_{\mu\nu}dx^\mu dx^\nu$ the line element in Einstein units, the corresponding line element in any other units (in particular atomic) will be written as $ds = \beta^{-1}(x)d\bar{s}$.

Our first task, that of writing Einstein equations in a scale-independent way, is achieved by performing the above conformal or scale transformation on Einstein equations. Once the generalized dynamic equations are derived, a wide range of phenomena will be analyzed and it will be found that no contradiction exists so far with any well-established fact.

(1) *Einstein equations.*—A conformal transformation, of the type just mentioned, transforms the usual Einstein equations into[5]

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R+f_{\mu\nu}(\beta) = -8\pi G(\beta)T_{\mu\nu}(\beta)+\Lambda(\beta)g_{\mu\nu}, \tag{1}$$

where

$$\beta^2 f_{\mu\nu}(\beta) = 2\beta\beta_{\mu;\nu}-4\beta_\mu\beta_\nu-g_{\mu\nu}(\beta\beta_{;\lambda}{}^{\lambda}-\beta^\lambda\beta_\lambda). \tag{2}$$

For any scalar α, $\alpha_\mu \equiv \alpha_{,\mu}$. The mathematical operations from now on should be performed using $g_{\mu\nu}$. Clearly, the choice $\beta = \text{const}$ reduces Eq. (1) to the Einstein gauge with $G(\beta)=\bar{G}=\text{const}$. Since the left-hand side of Eq. (1) is scale invariant, it does not change under a new scale transformation $ds \to ds' = \gamma(x)ds$; and since we want the entire Eq. (1) to be valid in any system of units, the right-hand side must also be scale invariant, and so we must have $\bar{G}\bar{T}_{\mu\nu} = GT_{\mu\nu}$. By use of the perfect-fluid approximation for $T_{\mu\nu}$ and the fact that $\bar{U}_\mu = \beta U_\mu$, it follows that $G(\beta)\rho(\beta)$ must transform like β^2, no matter how ρ and G transform separately.

(2) *Geodesic equations.*—Upon transforming the geodesic equation in general relativity via $d\bar{s} = \beta ds$, one obtains the scale-covariant geodesic equation[5]

$$\frac{d\xi^\mu}{d\lambda}+\Gamma_{\alpha\beta}{}^{\mu}\xi^\alpha\xi^\beta = \frac{\beta_{,\nu}}{\beta}(\epsilon g^{\mu\nu}-\xi^\mu\xi^\nu); \qquad \xi^\mu = dx^\mu/d\lambda, \quad \epsilon = g_{\mu\lambda}\xi^\mu\xi^\lambda. \tag{3}$$

For photons, $\epsilon=0$; for massive particles $\epsilon=1$, and $d\lambda = ds$.

(3) *Red shift.*—Since the frequency is defined as $u_\mu k^\mu$, where u_μ is the comoving velocity and k^μ is the photon wave vector, it is easy to derive from Eq. (3) that the following relation holds in any system of units: $\beta^{\Pi}\nu R = \text{const}$, where $R(t)$ is the expansion factor in the Robertson-Walker (RW) metric and $\Pi = \Pi(\kappa_\mu)$ is the power of the wave vector κ_μ under scale transformation.

(4) *Conservation laws.*—Using the perfect-fluid approximation for $T_{\mu\nu}$, we obtain from (1) the following conservation laws: Energy,

$$\dot{\rho}+(\rho+p)u_{;\mu}{}^{\mu} = -\rho\frac{d(G\beta)/dt}{G\beta}-3p\frac{\dot{\beta}}{\beta}; \tag{4}$$

momentum,

$$(p+\rho)\dot{u}^\mu = (g^{\mu\nu}-u^\mu u^\nu)\times[p_{,\nu}+pG_{,\nu}/G+(\rho-p)\beta_{,\nu}/\beta]; \tag{5}$$

baryon number,

$$(\mathfrak{M}u^\mu)_{;\mu} = \{\pi(G)-1\}\mathfrak{M}\dot{\beta}/\beta, \quad \mathfrak{M} = mn, \tag{6}$$

where n is the baryonic number density and $\pi(G)$ is the power of G: $G(\beta)=\bar{G}\beta^{-\pi(G)}$. So far $\pi(G)$ is unknown.

(5) *Cosmology.*—For any equation of state of the form $p = c_s{}^2\rho$, Eq. (4) is exactly solvable. The results for matter ($c_s{}^2=0$) and radiation ($c_s{}^2=\frac{1}{3}$) read in the RW metric

$$G(\beta)\rho_\gamma \propto \beta^{-2}(t)R^{-4}(t), \quad G(\beta)\rho_m \propto \beta^{-1}(t)R^{-3}(t). \tag{7}$$

The use of the RW metric, the energy-momentum tensor for a perfect fluid, and Eq. (7) leads to the following cosmological solutions ($3A = 8\pi\bar{G}\rho_0$): For $k=0$,

$$\beta(t)R(t)/R_0 = [1-\tfrac{3}{2}A^{1/2}f(t,t_0)]^{2/3}, \qquad f(t,t_0) \equiv \int_t^{t_0}\beta(t)dt; \tag{8}$$

for $k=-1$,

$$\beta(t)R(t)/R_0=(A/2B)(\cosh\psi-1),$$
$$B\equiv|k|/R_0^2, \qquad (9)$$
$$f(t,t_0)=(A/2B^{3/2})(\psi-\sinh\psi+C).$$

Here C is obtained by putting $t=t_0$, and $\psi=\psi^*$; ψ^* is then given by the first equation with $\beta(t)R(t)=R_0$. The case $k=+1$ is obtained by substituting $\psi=-i\theta$.

(6) *The three fundamental tests.*—By use of the geodesic equation (3), it can be shown that the formulas for the advance of perihelia and deflection of light rays are the same as in general relativity. The scale factor $\beta(t)$, however, influences the radar-echo-delay analysis, which can be shown to yield for the maximum round-trip delay the following new expression:

$$\beta(\Delta t)_{\max}=(\Delta t)_{\max}{}^0-8(M_sG)_0\ln[\beta R_s(t)/R_0], \qquad (10)$$

where the index zero corresponds to today and $R_s=R_\odot$.

(7) *Planetary orbits.*—The equations for two orbiting planets can be analyzed with Eq. (3). The significant result for the change of the period n can be written in the following form,

$$\dot{G}/G=h(\dot{n}_a-\dot{n}_t)/n, \qquad (11)$$

where $\dot{n}_a$ and $\dot{n}_t$ are the atomic and tidal contributions, and where the constant h is $\frac{1}{2}$ for the case $M=$const, $\beta=1$, $G\sim t^{-1}$, and +1 for the present theory, with $M\sim t^2$, $\beta\sim 1/t$, $G\sim 1/t$.

(8) *Cosmological constant,* Λ.—As seen from the previous treatment, the cosmological constant in the present theory is not constant at all. In fact it must scale like β^2. It is well known[2,6] that within the gauge fields with broken symmetries, a cosmological constant can be computed with a value of 10^{-6} cm^{-2}, whereas it is known that its value today cannot exceed 10^{-57} cm^{-2}. Since, however, 10^{-6} is computed at $T=200$ GeV, i.e., at 10^{-10} sec, a scaling factor β^2, with $\beta=t_0/t$, is needed to bring 10^{-6} down to 10^{-60} at $t=t_0\sim 10^{17}$ sec. A more formal proof will be given at the end [see Eq. (12)].

(9) *Determination of* $\beta(t)$.—The comparison of the present theory with observations is dependent upon the function $\beta(t)$. Since $\beta(t)$ represents the freedom we have in choosing the system of units, it cannot be determined from within the theory, i.e., no dynamical equations can exist for $\beta(t)$, and external conditions must be imposed. The connection with gauge fields and the cosmological constant already suggest $\beta\sim 1/t$. As an alternative possibility, we shall also adopt the Dirac large-number hypothesis,[4,5] which states that $G\sim 1/t$ and $M\sim t^2$. From the present theory we have $G(\beta)\sim\beta^{-\pi(G)}$ and $M\propto\beta^{\pi(G)-1}$ [Eq. (6)]. Solving for β we obtain $\beta\sim 1/t$, i.e., $G(\beta)\sim\beta$, $M\sim\beta^{-2}$, $MG\beta\sim$ const. We shall therefore write $\beta=t_0/t$, where t_0 is the age of the Universe today.

(10) *Comparison with observations.*—Let us first analyze Eq. (11). The most recent determinations[7] are $\dot{n}_a=-36\pm 5.0$ sec/century^2, $\dot{n}_t=-26.0\pm 2.0$ sec/century^2, and $n=17.33\times 10^8$ sec/century from timings of occultations of stars by the moon. Since $-\dot{G}/G=1/t_0=H_0$ and the most recent determinations of H_0 yield $H_0=55\pm 7$ km sec^{-1} Mpc^{-1}, the left-hand side of Eq. (11) is $(5.6\pm 0.7)\times 10^{-11}$ yr^{-1}. If $\dot{n}_a-\dot{n}_t$ is taken to be a nonzero result, then the present theory with $G\sim 1/t$, $M\sim t^2$, $\beta\sim 1/t$, and $h=1$ is favored over the so-called primitive theory, $G\sim 1/t$ with M and β being constant. In fact, using $n=17.33''\times 10^8$ century^{-1}, the right-hand side of (11) is

$$h\times(5.8\pm 3.1)\times 10^{-11}/\text{yr}$$

which is in good agreement with $\dot{G}/G$ if $h=1$. The choice $h=1$ looks preferable over $h=\frac{1}{2}$.

The three fundamental tests have already been discussed and the fact that we obtain the same result as in ordinary theory is most comforting, since other theories have failed at precisely that level.

For large cosmological times, the value of the deceleration parameter q_0 turns out to be very close to zero. This is in agreement with the most recent determination of q_0 from observational data.[8] Several other tests have been performed by the authors as well as by others (using the present mathematical formulation) regarding the magnitude versus red-shift relation, angular diameters (both isophotal and metric) for giant elliptical galaxies,[5,9] stellar evolutionary effects,[10] etc. All the tests performed so far have revealed that the present theory fares with observations either equally well as ordinary cosmology or slightly better.

A final important consideration is in order. The theory just presented is entirely classical, i.e., it can be applied only to macroscopic objects: A corresponding theory valid for low quantum numbers has yet to be constructed. A comparison with manifestly gauge-invariant theories is therefore not immediate. However, an interesting connection must exist. In fact, our scale function β plays a role analogous to the Higgs field φ, as one can see upon comparing the action

$$I=\int g^{1/2}d^4x\{-\beta^2R+\alpha\beta^4+6\beta_\mu\beta^\mu+L_m\},$$

used to derive Eq. (1), with the usual double-humped Higgs Lagrangian.[2] The $-\mu^2$ term is replaced by the contracted Ricci tensor and the $\lambda\varphi^4$ term by $\alpha\beta^4$. The comparison can also be made at the level of the dynamic equations, if one considers that the additional term in Eq. (1), i.e., $\beta^2 f_{\mu\nu}$, is just the so-called improved energy-momentum tensor for scalar particles.[11] Furthermore, from the work of Linde, Weinberg, and Jackiw[1-3] (see also Canuto and Lee[6]) we know that

$$\sigma^2\equiv\langle\varphi\rangle^2\sim T^2,\quad \Lambda\sim\sigma^4\sim T^4. \tag{12}$$

For a relativistic gas $T^2\sim t^{-1}$, and so it follows that $\Lambda\sim t^{-2}$. Since in our theory $\Lambda\sim\beta^2$, we can conclude that $\beta\sim 1/t$.

We would like to note that the connection with gauge fields has provided another interpretation to the scale-invariant theory besides the one presented originally in Ref. 5. At the same time, it has also provided the way to determine the scale factor $\beta(t)$ via the cosmological constant Λ. Since Λ is a function of both G and microscopic constants, $\Lambda^{-1/2}$ can be thought of as the fundamental length that relates microscopic physics and gravitation. We believe that this analysis has contributed one step further in the understanding of gravitation in the light of gauge fields and broken symmetries.

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