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Non-Abelian Gauge Fields and Nonrelativistic Bound States

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The second-quantized Coulomb-gauge Hamiltonian for any nonrelativistic system of fermions minimally coupled to non-Abelian gauge fields is derived by performing successive non-Abelian Foldy-Wouthuysen transformations. In this general formalism, a detailed analysis is made of threshold fermion-antifermion bound states in the weak-coupling limit and the form of the two-loop nonrelativistic Bethe-Salpeter kernel is determined. This kernel gives a static potential independent of the fermion mass, but only for singlet states of the gauge group.

The nonrelativistic limit of the strong interactions may be sufficiently simple to give a more tractable methematical description than the fully relativistic theory. Furthermore, the phenomenological successes of nonrelativistic potential models[1] in describing the J/ψ family of mesons provide strong evidence that there exist physical processes dominated by the nonrelativistic strong interactions. This Letter is a first attempt to develop a consistent general formalism for systems with large quark masses, or, more generally, for any system of nonrelativistic fermions. The dynamical model is that of massless non-Abelian gauge fields minimally coupled to fermions (quantum chromodynamics). As an example of the usefulness of this formalism, fermion-antifermion bound states are discussed in perturbation theory through two loops.

The Lagrangian in first-order form is

$$L=\int d^3x\left[\tfrac{1}{4}F_{\mu\nu}{}^{a}F_{a}{}^{\mu\nu}-\tfrac{1}{2}F_{\mu\nu}{}^{a}(\partial^{\mu}A_{a}{}^{\nu}-\partial^{\nu}A_{a}{}^{\mu}+gt_{abc}A_{b}{}^{\mu}A_{c}{}^{\nu})+\bar{\psi}(i\partial\!\!\!/-gt^{a}A\!\!\!/^{a})\psi-m\bar{\psi}\psi\right]. \tag{1}$$

Fermions are in the fundamental representation of the group. The natural gauge condition for nonrelativistic systems is the radiation gauge, $\vec{\nabla}\cdot\vec{A}^{a}=0$, as used by Schwinger,[2] who first quantized this model. Also, the first-order formalism is advantageous since it does not possess any ghosts in the radiation gauge. Imposing this gauge and transforming to the Hamiltonian gives

$$H=\int d^3x\left[\tfrac{1}{2}(\vec{E}^{a}\cdot\vec{E}^{a}+\vec{B}^{a}\cdot\vec{B}^{a})+\psi^{\dagger}(-i\vec{\alpha}\cdot\nabla+\beta m)\psi-g\psi^{\dagger}\vec{\alpha}t^{a}\psi\cdot\vec{A}^{a}+\tfrac{1}{2}(\nabla\varphi_{a})\cdot(\nabla\varphi_{a})\right], \tag{2}$$

where $\vec{E}^{a}$ ($\nabla\cdot\vec{E}^{a}=0$) is the conjugate variable to $\vec{A}^{a}$, $B_{i}{}^{a}=\tfrac{1}{2}\epsilon_{ijk}F_{jk}{}^{a}$, and φ_{a} is the non-Abelian generalization of the Coulomb potential, $\varphi_{a}=\int d^3y\,D^{ab}(x,y)j_{b}{}^{0}(y)$. $D^{ab}(x,y)$ is an integral operator defined by

$$[\nabla^{2}\delta_{ab}+gf_{adb}\vec{A}_{d}\cdot\vec{\nabla}]D^{bc}(x,y)=\delta^{ac}\delta^{3}(x-y) \tag{3}$$

and $j_{a}{}^{0}(x)$ is the non-Abelian charge density given by

$$j_{a}{}^{0}(x)=gf_{abc}\vec{E}_{b}\cdot\vec{A}_{c}+g\psi^{\dagger}t^{a}\psi; \tag{4}$$

$Q_{a}=\int d^3x\,j_{a}{}^{0}(x)$ are the non-Abelian charges, generators of the gauge group. To proceed to the nonrela-

tivistic limit the Hamiltonian, Eq. (2), is subjected to a succession of non-Abelian Foldy-Wouthuysen transformations[3] to obtain the formal m^{-1} expansion of H through order m^{-1}. The transformed Hamiltonian is

$$H' = m\int d^3x\,\psi^\dagger(x)\beta\psi(x) + \int d^3x\{\tfrac{1}{2}[\vec{E}_a\cdot\vec{E}_a + \vec{B}_a\cdot\vec{B}_a] + \int d^3y\,d^3z\, j_a^0(x)D_{ab}(x,y)\nabla_y^2 D_{bc}(x,y)j_c^0(z)\}$$
$$+ m^{-1}\int d^3x\,\tfrac{1}{2}[\psi^\dagger(x)\beta(i\nabla + gt^a\vec{A}^a)^2\psi(x) + \psi^\dagger(x)\beta\vec{\sigma}gt^a\psi(x)\cdot\vec{B}_a] \quad (5)$$

where $D_{ab}(x,y) = [(\nabla^2 + g\vec{A}\cdot\nabla)^{-1}]_{ab}$. The non-Abelian static Coulomb interaction

$$\int d^3x\,d^3y\,d^3z\{j_a^0(x)D_{ab}(x,y)\nabla_y^2 D_{bc}(x,y)j_c^0(z)\}$$

has a universal form [$j_a^0(x)$ depending on the type of particle interacting with the gauge fields.] However, the formal expansion in Eq. (5) may fail even if external momenta are restricted to nonrelativistic values since loop integrals could be dominated by ultraviolet regions of integration in such a way as to invalidate the m^{-1} expansion. The absence of such behavior in suitably renormalized matrix elements is guaranteed, at least within perturbation theory, by the following theorem,[3] an extension of the Appelquist-Carrazone theorem[4]: For any renormalized Green's function in which all external fermion momenta are in the nonrelativistic domain and in which external gluon momenta are small compared to the fermion mass, fermion pair creation is negligible, fermions couple only to Coulomb propagators, and all momentum transfers from fermion lines are effectively small; that is, loops involving these momenta are infrared dominated, and large contributions come only from the region in momentum space $\lesssim \beta m$ ($\beta = v/c$). Therefore, in the m^{-1} expansion, terms dropped will contribute in any order of perturbation theory a relative amount of order β.

The Hamiltonian in Eq. (5) is the starting point for a nonrelativistic calculation of threshold fermion-antifermion singlet bound states. Their threshold character[5] is imposed by requiring that the binding energy is the same order of magnitude as the fermion kinetic energy; that is, $p^2/2m \sim \beta^2 m$. The self-consistency of this requirement must be checked by explicit calculations. The nonrelativistic kernel, which is the Fourier transform of the nonrelativistic Schrödinger potential term if the kernel has a static limit, consists of two-fermion irreducible graphs, initial-state self-energy corrections, and parts of two-fermion reducible graphs which are not iterations of the other two types of contributions to the kernel. The existence of this last type of contribution is seen by considering iterations of the fully relativistic kernel, which does not contain two-fermion reducible graphs. Since there may be terms in the nonrelativistic limit of iterations of this full kernel not contained in iterations of the nonrelativistic kernel, any such term[6] must be considered part of the nonrelativistic kernel itself. If ΔT is the time scale over which the kernel acts, and $\epsilon \sim \beta^2 m$ is a typical amount the fermions are off shell in the bound state, then an instantaneous potential is obtained when $\Delta T \ll \epsilon^{-1}$, that is, the kernel must "see" the quarks as free and essentially at rest. Therefore, to obtain a static limit, dominant values of the loop momenta must be large compared to ϵ (but small compared to m), so that ϵ may be set to zero in intermediate energy denominators of old-fashioned perturbation theory. This requirement is satisfied explicitly for the sum of graphs through two loops, although not true graph by graph.

The appropriate kinematic region of the kernel is $p_i^2 - m^2$, t, and $s - 4m^2$ all of order $\beta^2 m^2 \ll m^2$ (s and t are the Mandelstam variables and $\{p_i^2\}$ are the external fermion momenta). In lowest-order perturbation theory the only contribution to the kernel is from a single Coulomb exchange. The initial and final states are assumed to be singlet states with respect to the internal symmetry.

At the one-loop level, only the two graphs shown in Fig. 1 contribute. The dashed lines represent instantaneous Coulomb exchanges and all graphs are written in old-fashioned perturbation theory with time increasing from left to right. In first-order form, the vector-meson loop in Fig. 1(b) represents the sum of two contributions; the first is an $\vec{A}^a$ propagator and an $\vec{E}^a$ propagator, and the second is two mixed $\vec{A}^a$-$\vec{E}^a$ propagators. Other time orderings must be included where necessary. These graphs in Fig. 1 add to give the following known result[7] for the kernel to one loop:

$$K_1 = [-ig^2(-t)/-t]\,T_2 N, \quad (6)$$

where $T_2 = \sum_a t^a t^a$, N is the dimension of the fundamental representation of the group, and the diagonal spin indices are suppressed. $g^2(-t)$ is the effective coupling constant at $-t$ for a pure Yang-Mills theory. To this order, coupling-constant

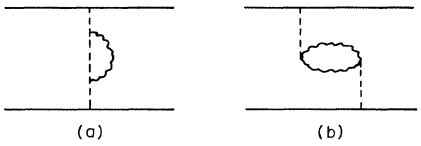

FIG. 1. One-loop contribution to the nonrelativistic kernel.

renormalization is due entirely to Coulomb-propagator corrections which nonrelativistically involve only the Yang-Mills part of the theory. Also the fermion self-energy renormalization and the fermion-Coulomb vertex renormalization are exactly equal but this strict equality fails in higher orders. A precise general statement is given below.

In two loops, there are graphs which contribute to the kernel in addition to the Coulomb-propagator corrections which give the two-loop contribution to the Yang-Mills β function expressing $g^2(-t)$ in terms of $g^2(\mu)$ (μ is the renormalization point for the Yang-Mills fields, chosen such that $\mu < m$). These additional graphs are shown in Fig. 2. The initial and final states are off energy shell by amounts ϵ_1 and ϵ_2, respectively, with $\epsilon_1, \epsilon_2 \sim \beta^2 m$. Similar graphs occur with the fermion and antifermion lines interchanged. Figure 2(f) is an external fermion self-energy correction, which, on one side, is included in the kernel. Figure 2(g) represents a more subtle contribution, which is two-fermion reducible. It is the combination of a nonrelativistically negligible part of the one-loop kernel and the single Coulomb exchange. To expose this contribution consider the energy denominator (in old-fashioned perturbation theory in Fig. 1(b):

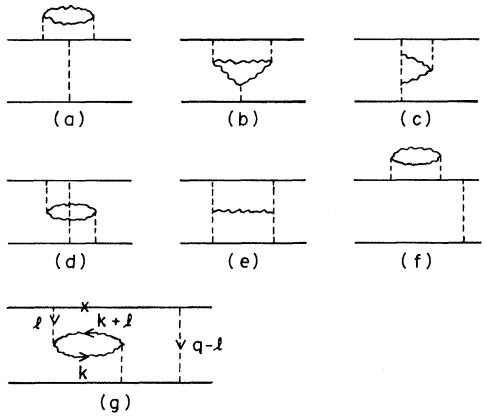

FIG. 2. All two-loop contributions to the nonrelativistic kernel except Coulomb-line corrections.

$$(k+|\vec{l}-\vec{k}|+\epsilon_1)^{-1} = (k+|\vec{l}-\vec{k}|)^{-1} - (k+|\vec{l}-\vec{k}|)^{-1}\epsilon_1(k+|\vec{l}-\vec{k}|+\epsilon_1)^{-1}. \quad (7)$$

The second term on the right-hand side is a relativistic correction ($\epsilon_1 \sim \beta^2 m$, not βm) correctly ignored at the one-loop level. However, in graph 2(g) the energy denominator of the two-fermion intermediate state is ϵ_1^{-1}, and therefore the second term in Eq. (7) gives a nonnegligible contribution to the two-loop nonrelativistic kernel. The X on the fermion line indicates that the second term in Eq. (7) is to be used for the appropriate energy denominator. This term, therefore, is not part of the iteration of the nonrelativistic kernel, which involves only the first term on the right-hand side of Eq. (7), and moreover, it is of the same order in β as the other graphs in Fig. 2.

The logarithmic dependence on the fermion renormalization point of graph 2(f), a self-energy correction, cancels with the renormalization point dependence of the vertex correction graphs 2(a) and 2(b). The necessity of this cancellation follows from the fact that as $q \to 0$ the bare instantaneous part of the Coulomb propagator couples only to the charge operator [see Eq. (5)]. Moreover, charge conservation implies,[3] in a manner analogous to quantum electrodynamics, that the fermion self-energy renormalization is exactly equal to the renormalization of the fermion-charge vertex defined by the fermion-Coulomb (proper) vertex with the external instantaneous Coulomb line, at zero momentum, coupled only to j_a^0. The above cancellation is an explicit example of this Ward identity in two loops. Furthermore, the Ward identity implies[3] that the divergent part of the fermion-Coulomb vertex is equal to the divergent part of the fermion self-energy. No unrenormalized vertex graph which is not part of the fermion-charge vertex will contain primitive divergences, as, for example, the unrenormalized finite graph 2(c). Also, coupling-constant renormalization results only from Coulomb-propagator corrections, which, nonrelativistically, are in the pure Yang-Mills sector. Furthermore, there is a cancellation of the infrared ϵ logarithmic-dependent terms between graphs 2(a) and 2(d), and also between graphs 2(f) and 2(g). In each pair, what distinguishes the two graphs

is whether the Coulomb line is absorbed by the fermion or by the antifermion. In the infrared limit of this Coulomb momentum, which produces the infrared logarithmic dependencies, the sum of the distinguished Coulomb lines in each of the two graphs measures the total charge in the initial state, which is zero since the initial state is a singlet. Adding all the graphs in Fig. 2 gives for the exact form of the kernel through two loops

$$K_2 = \frac{-ig^2(-t)}{-t} T_2 N\left[1 + bc_2{}^2\left(\frac{g^2(-t)}{16\pi^2}\right)^2\right] \qquad (8)$$

($c_2\delta_{ab} = f_{acd}f_{bcd}$), where b is a numerical constant. This kernel represents a static nonrelativistic potential which is valid in the infinite-quark-mass limit. Note that the static potential in the nonrelativistic limit through two loops is obtained if, and only if, the initial and final states are singlets. At the three-loop level, there are graphs in which $\ln\epsilon$ dependence arises from vanishing transverse gluon momentum [e.g., the graph obtained from Fig. 2(e) by adding, in the middle of the graph, a bare Coulomb exchange between the fermion and antifermion]. The cancellation of this type of dependence is still unresolved. However, the simplicity of Eq. (8) and the fact that the static form depends only on charge conservation suggest that to all orders of perturbation theory there is a well-defined static limit for the kernel in singlet states.

In conclusion, even though the weak-coupling nonrelativistic kernel has qualitatively the same form as in quantum electrodynamics through two loops, this new result is important for several reasons. It illustrates the usefulness of the nonrelativistic formalism, by making obvious the physical bases of the results. The kernel possesses a well-defined nonrelativistic limit free of divergences, in which all logarithmic dependence on ϵ_1 and ϵ_2 explicitly has canceled, but only for group singlet states. The coupling constant which appears is the effective Yang-Mills coupling (defined in the Coulomb gauge) at a momentum value equal to the momentum transfer, so that it is the effective coupling constant relevant for formation of the bound state. Treatment of baryons is similar to the above analysis and does not present any new difficulties. In addition, there are many possibilities for nonperturbative approaches. For example, it may be worthwhile to consider the static Coulomb energy term directly or to try to find semiclassical solutions for the Hamiltonian in Eq. (5).

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Note added.—A similar analysis in the Lagrangian formalism through two loops has been done by T. Appelquist, M. Dine, and I. Muzinich, to be published. I would like to thank them for several interesting discussions.

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