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## Effective-Potential Approach to Graviton Production in the Early Universe


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The production of gravitons in a homogeneous, isotropic, spatially flat universe containing radiation and a small admixture of cold baryons is studied by an effective-potential method which takes into account the back reaction of the produced gravitons on the geometry. The total probability per unit volume to produce a pair of gravitons is finite and has a very small value for a universe with parameters corresponding to my own.


In the early universe the process of pair creation may play a significant dynamical role.[1] Field-theoretic calculations of this process encounter not only the usual ultraviolet divergences of flat-space field theory but also divergences arising from the cosmological space-time singularity. Typical attempts to resolve these latter difficulties and obtain a finite number of produced particles have involved either limiting the calculation to a space-time region which does not include the singularity or altering the geometry to remove the singularity in an *ad hoc* way.

In this Letter I shall outline an effective-potential approach to the calculation of pair creation in the early universe in the context of a particular model problem. The problem is the production of gravitons in a homogeneous, isotropic, spatially flat universe containing classical radiation and a small admixture of cold classical baryons. This effective-potential method takes into account the back reaction of the produced gravitons on the metric. If it is assumed that no gravitons are produced for a universe containing radiation alone, we shall find that for a universe with a small admixture of cold baryons this back reaction is just such as to regulate the behavior of the classical geometry at the singularity so as to make the total graviton pair production probability per unit volume finite.

I shall calculate the amplitude $\langle 0_+|0_-\rangle$ that the no-graviton vacuum at early times remains the no-graviton vacuum at late times. My fundamental starting point is the expression for this amplitude in the presence of classical sources whose stress-energy tensor density is $\mathcal{J}_{\alpha\beta}$. In terms of a Feynman sum over fields[2]

$$\langle 0_+|0_-\rangle_{\mathcal{J}} = \int \delta g \exp\{i(S[g] - \int d^4x\, \mathcal{J})\} \equiv \exp(iW[\mathcal{J}]). \qquad (1)$$

where $S = S_0[g] + \int d^4x\, \Lambda$, $S_0[g] = -l^{-2}\int d^4x(-g)^{1/2}R$ + (surface terms) being the gravitational action, $\Lambda$ the Lagrangian of the matter, $\mathcal{J} = \mathcal{J}_\alpha{}^\alpha$, and $l = (16\pi G)^{1/2} = 1.2\times10^{-32}$ cm is the Planck length. Throughout I use units with $\hbar = c = 1$ and signature $+2$, and for compactness will frequently suppress tensor indices. The sum in (1) is over all geometries between the initial singularity and a late-time spacelike surface. The classical geometry $g(x)$ is defined in terms of the metric field operator $\hat{g}(x)$ by $g(x) = \langle 0_+|\hat{g}(x)|0_-\rangle_{\mathcal{J}}/\langle 0_+|0_-\rangle_{\mathcal{J}}$ and in the presence of particle production will be a complex quantity. The effective potential $\Gamma[g]$ is the Legendre transform of $W[\mathcal{J}]$ with respect to the source $\mathcal{J}$, $\Gamma[g] = W[\mathcal{J}] + \int d^4x\, \mathcal{J}$. Its importance arises from the fact that the classical geometry $g$ satisfies the variational equation

$$\delta\{\Gamma[g] - \int d^4x\, \mathcal{J}\}/\delta(x) = 0. \qquad (2)$$

The vacuum persistence amplitude in the absence of external sources is then given by

$$\langle 0_+|0_-\rangle = \exp(i\Gamma[g]), \tag{3}$$

evaluated at the solution to (2) with $\mathcal{J}=0$. Equation (2) may be regarded as *defining* the possible vacuum states and Eq. (3) gives their persistence amplitudes.

I will now search for the solutions to (2) in which the classical geometry is homogeneous, isotropic, and spatially flat and which further have an initial singularity and which at late times approach a real solution of Einstein's equations. The metric then has the form

$$ds^2 = a^2(\eta)(-d\eta^2 + dx^2 + dy^2 + dz^2). \tag{4}$$

For the matter we shall take a mixture of radiation for which the trace of the stress-energy tensor satisfies $T=0$ and baryons for which $T=-\tilde{\rho}_b/a^3$, where $\tilde{\rho}_b$ is a positive constant.

Three approximations will be used in evaluating $\Gamma[a]$ for metrics of the form of (4). First, the sum over fields in (1) will be restricted to the pure gravitational wave modes. The matter content of the universe is being treated classically and it is inappropriate to quantize the density and vorticity perturbations associated with these degrees of freedom. This restriction is most easily implemented by writing the integration variables in (1) as $g=a^2\eta+h$, where $\eta$ is the Minkowski metric and then integrating only over tensors $h(x)$ which are time orthogonal, transverse, and traceless.[3] Second, the effective potential, $\Gamma[a]$, will be calculated in the one-loop approximation or equivalently $W[\mathcal{J}]$ will be calculated by the method of steepest descents. Then,[2] when $\mathcal{J}=0$

$$\Gamma[a] = S[a] + \frac{i}{2}\,\mathrm{Tr}\ln\left[\frac{\delta^2 S}{\delta h(x)\delta h(x')}\right]. \tag{5}$$

This approximation is appropriate because in an expansion of $\Gamma[a]$ in powers of $\hbar$ the single loops are the next term after the classical one whose difficulties I have discussed and also because the single loops contain the simplets diagrams corresponding to the creation and annihilation of a graviton pair—the process under consideration.

In (5), the second term may be reexpressed as $-\frac{1}{2}i\,\mathrm{Tr}\ln[G(x,x')]$, where $G(x,x')$ is the Green's function of the linear wave equation $\delta S_0/\delta h=0$ for gravitational waves propagating in the background geometry of (4). Boundary conditions must be fixed for this Green's function. In an important paper[4] which underlies the present work Grischuk has shown one way to do this. The simplest case is when the matter is pure radiation, $\tilde{\rho}_b=0$. Then Grischuk shows that *exact* solutions to the wave equation have the space-time dependence $h(x) \propto [1/a(\eta)]\exp(\pm i\omega\eta + \vec{k}\cdot\vec{x})$ with $\omega=|\vec{k}|$ so that in this sense positive- and negative-frequency solutions can be defined over the whole of space-time and do not mix. Equivalently, when expressed in terms of the variables $p(x)=a(\eta)h(x)$, the quadratic part of the expansion of the Lagrangian $S$ in $p(x)$ has the form of the Lagrangian for free gravitational waves in flat space-time. It is therefore natural to demand that the vacuum be annihilated by the positive-frequency parts of the field $p(x)$ or equivalently that the Green's function $G(x,x')$ be the flat-space Feynman Green's function times a conformal factor. Since the theory in the pure-radiation case is then essentially equivalent to free-field theory in flat space-time, there will be no graviton production, the second term in (5) will be a constant, the extremum of $\Gamma(a)$ will satisfy Einstein's equations, $\Gamma(a)$ will be real, and the vacuum will persist, $|\langle 0_+|0_-\rangle|=1$.

If baryons are now added to the matter ($\tilde{\rho}_b\neq 0$), the same boundary conditions may be applied to $G(x,x')$ but $\Gamma[a]$ will no longer be real. In general, it would be impossible to obtain $\Gamma[a]$ as a closed-form functional of $a(\eta)$ but it can be computed perturbatively in closed form if $\tilde{\rho}_b$ is sufficiently small. The constant dimensionless parameter governing such an expansion is $\xi = l\rho_b/\rho_r^{3/4}$, where $\rho_r$ is the energy density in radiation. Since $\xi \sim 10^{-27}$ for our universe, an expansion in $\xi$ should give an excellent approximation to $\Gamma[a]$. Indeed, for present baryon masses, $\xi$ is very small even for universes for which the ratio of baryons to radiation quanta is quite large. My third approximation in computing $\Gamma[a]$ is to evaluate it to second order in $\xi$.

To evaluate $\Gamma[a]$ in the one-loop approximation the action $S_0$ is first expanded to quadratic order in $h$. Using the field equations to eliminate terms which are of nominally lower order but actually higher order in $h$, working in the transverse, traceless, time-orthogonal gauge, and using the variables $p=ah$, one finds

$$S[g] = \int d^4x\{6(a')^2/l^2 + \tfrac{1}{4}p_i{}^j[\Box^2 p_j{}^i - A(\eta)p_j{}^i]\}. \tag{6}$$

Here, a prime denotes a derivative with respect to $\eta$, $A=a''/a=6a^2R$, and $\Box^2=\eta^{\alpha\beta}\partial_\alpha\partial_\beta$ is the flat-space wave operator.

The quantity $A$ vanishes when $\xi=0$; so for small $\xi$, $A$ will be small. The action in (6) has the form of a flat-space action for the fields $p_i{}^j$ plus a quadratically coupled perturbing potential $A(\eta)$. The effective potential in the one-loop approximation can be evaluated as a series in $A$ using the usual flat-space Feynman rules. The leading terms are infinite and must be regularized. To do this I use the dimensional regularization procedure,[5] continuing in the number of conformally related flat-space dimensions. In this procedure the loop which is first order in $A$ vanishes. The second-order loop has a logarithmic divergence which cannot be removed by a renormalization of $G$ or the cosmological constant. This nonrenormalizability is familiar from other examples.[6] To proceed, I subtract out this divergence by adding to the action a counter term whose divergent part is

$$S_c[a]=(72\epsilon)^{-1}\int d^4x(-g)^{1/2}R^2, \tag{7}$$

where $\epsilon=8\pi^2(n-4)$, $n$ being the number of space-time dimensions. The resulting effective potential will be proportional to the total spatial-coordinate volume because of spatial translation invariance. Denoting this by $V$ we find

$$V^{-1}\Gamma[a]=6\int_0^\infty d\eta(a')^2/l^2+(64\pi^2)^{-1}\int_0^\infty d\eta\int_0^\infty d\eta' A(\eta)K(\eta-\eta')A(\eta'), \tag{8}$$

where the singularity $a=0$ has been located at $\eta=0$. The kernel $K$ is given by

$$K(\eta)=i\delta(\eta)+(2/\pi)\int_0^\infty d\omega\cos(\omega\eta)\ln(\omega/\omega_0). \tag{9}$$

The parameter $\omega_0$ is arbitrary and arises because the real finite part of the counter term of the form (7) is unspecified. This arbitrariness is yet another reflection of the nonrenormalizability of the theory.

The variational equation (2) which determines the classical geometry from this $\Gamma[a]$ is a fourth-order, nonlocal, one-dimensional integrodifferential equation for $a(\eta)$. This equation has an integral which is

$$\tilde{\rho}_r=6(a')^2/l^2-a\tilde{\rho}_b+(64\pi^2)^{-1}[-2a'(I/a)'+(a''/a)I], \tag{10}$$

where $\tilde{\rho}_r$ is constant and

$$I(\eta)=\int_0^\infty d\eta' K(\eta-\eta')[a''(\eta')/a(\eta')]. \tag{11}$$

This equation is to be solved with the boundary conditions that $a=0$ at $\eta=0$ (the singularity) and that at large $\eta$ it approach the real solution of Einstein's equations

$$a(\eta)=(\tilde{\rho}_r/6)^{1/2}l\eta+(\tilde{\rho}_b/24)(l\eta)^2. \tag{12}$$

I have analyzed the solutions of this problem. Especially instructive is the local approximation in which only the $\delta$-function part of $K(\eta)$ is retained. Equation (10) can then be reduced to a second-order nonlinear differential equation[7] for $(a')^{2/3}$ as a function of $a$. For sufficiently small $\eta$, this equation can be linearized in $\xi$, solved in terms of Bessel functions, and matched onto (12) at larger $\eta$. The details of this are too complex to be given here but the important result is that near the singularity instead of the behavior of (12) the effective potential leads to a small-$\eta$ behavior for $a(\eta)$ of the form

$$a(\eta)=(\tilde{\rho}_r/6)^{1/2}l\eta+\text{const}\times\eta^4\cdots. \tag{13}$$

If the nonlocal term in $K$ is included the behavior of the next term after the linear one is modified to const$\times\eta^4/\ln\eta$. This small-$\eta$ behavior means that the effective potential per unit spatial volume [Eq. (8)] evaluated at the classical geometry will have a finite imaginary part in contrast to the infinite value obtained when the solution to the classical Einstein equations alone [Eq. (12)] is used. Within the context of approximations used here, the back reaction on the metric has thus regulated the production of the gravitons near the singularity.

Putting the equations in dimensionless form, the magnitude of the vacuum persistence amplitude can be estimated. Stating this in terms of the physically more interesting total probability $P_b$ to produce a pair of gravitons in the spatial volume occupied by one baryon, I find $P_b=\alpha(\omega_0)\times m_p l$, where $\alpha$ is a dimensionless constant of moderate value and $m_p$ is the proton mass. Assuming closure density for the matter, $P_b\sim10^{-20}\alpha$ for the parameters of our universe so that the graviton production probability in this model is enormously small.

The effective-potential method for calculating particle production in cosmological models has several advantages. It takes the back reaction of the produced particles into account. Regularization can be accomplished in the action itself rather than in quantities of a more complicated tensorial character such as the stress-energy tensor. Finally, it lends itself naturally, as here, to approximation schemes which can be clearly related to the basic quantum-mechanical law for amplitudes, Eq. (1). It would be of great interest to apply this method to more general and more physically realistic cosmological models including anisotropies.

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# Picosecond Dephasing of Coherently Excited Vibrations in Liquid $N_2$-Ar Mixtures

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Vibrational-dephasing times have been measured in mixtures of liquid $N_2$ and Ar. The correlation time of the molecular motion initiating the dephasing process is calculated to be 2.2 and 4.6 ps in pure $N_2$ and pure Ar, respectively. These rather long correlation times suggest that the dephasing results from the average force field determined by the number of nearest-neighbor molecules.

Recently it has been shown that after a coherent excitation of molecular vibrations by stimulated Raman scattering (SRS), the dephasing of the vibrations can be probed directly by measuring the intensity of the coherent anti-Stokes Raman scattering as a function of the delay between excitation and probe pulse.[1,2] By this method the vibrational-dephasing time has been obtained for a number of pure liquids.[2-4]

The dephasing of the coherent excited ensemble is the result of a modulation of the vibrational transition frequencies by stochastic perturbations which arise from the intermolecular interactions of the excited molecules with their individual surroundings. The modulation itself is determined by the dynamics of the molecular motions in the liquid.

Several descriptions have been developed to relate the dephasing time to a vibrational correlation function.[2,5-7] Recently, Rothschild[7] gave an analysis in terms of molecular-dynamics (M.D.) calculations for liquid nitrogen, a Lennard-Jones liquid for which the thermodynamical properties are reasonably understood by M.D. calculations. Using the results of M.D. calculations for calculating the mean-square frequency displacement, a correlation time $\tau_c$ can be calculated in case of a fast modulation of the oscillator frequencies. By comparison with the M.D. calculations it may then be determined which type of molecular motion is responsible for the vibrational dephasing.

In order to investigate the influence of the molecular environment upon the dephasing of nitrogen, the dephasing times were measured in liquid