Stability of Superflow in ^{3}He-A

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The stability condition is derived for uniform bulk superflow along a uniform texture in ^{3}He-A. The weak-coupling free-energy parameters in the dipole-locked Ginzburg-Landau regime give stable flow, but a 10% departure from these values can give instability. When present, the instability can reflect the absence of any activation-energy barrier against decay of the superflow through the appearance of vortex textures near the surface. We relate these conclusions to the hydrodynamic instability reported by Hook and Hall.

In the A phase of superfluid helium-3, single-valuedness of the order parameter does *not* imply quantization of circulation: If the anisotropy axis $\vec{l}$ of the gap varies in space then[1]

$$\nabla\times\vec{v}_s=\tfrac{1}{2}\epsilon_{ijk}\,l_i\nabla l_j\times\nabla l_k\,. \qquad (1)$$

The circulation around a closed contour can therefore change continuously in time, provided the configuration of $\vec{l}(\vec{r})$ enclosed by the contour varies appropriately. At the surface of a vessel $\vec{l}$ is constrained to be along the normal[2] and the curl of $\vec{v}_s$ is fixed by surface geometry.[3] Thus circulation *is* quantized around closed surface contours, but not around contours in the bulk.

There are thus two unusual ways in which ^{3}He-A can dispose of a macroscopic current in a toroidal container: (a) Vortex textures—analogous to vortices in ^{4}He but with continuously distributed vorticity and no singular cores—can form and move across a supercurrent, thereby reducing the bulk (but not the surface) current without the nucleation of singular regions.[4] (b) Surface current can be reduced (with an accompanying reduction to bulk current) by the motion of a special type of surface point singularity (a "boojum").[5,6] The relative importance of these mechanisms depends on details of the pinning, nucleation, and equilibrium populations of boojums, as well as on the energetics of vortex texture formation against a background of superflow. We wish here to point out some rather remarkable features of the latter problem.

Unlike singular vortices in ^{4}He, vortex textures in ^{3}He-A can evolve continuously from configurations with no vorticity, and can begin to degrade the superflow even when they are far from fully developed. The question arises of whether this immediate reduction in flow energy may, from the start, outweigh the energies (specifically the bending energy in $\vec{l}$ and the energy of misalignment between $\vec{l}$ and $\vec{v}_s$) which must be spent in forming such textures. If this were so then the decay of uniform bulk superflow would require no activating fluctuations, being a consequence of the ordinary hydrodynamic equations: The supercurrent would not be "super" at all.

To explore this possibility we have tested the stability of an equilibrium configuration of uniform flow (along $\hat{z}$) between plane parallel walls (normal to $\hat{y}$) with a boundary condition of macroscopic periodicity along $\hat{z}$ to simulate the toroidal connectivity that stabilizes the superflow in ^{4}He. Near the walls $\vec{l}$ must turn toward the normal, but everywhere in the bulk it lies along $\hat{z}$. The macroscopic periodicity requires that in the absence of singularities in the $\vec{l}$ field, the mean $\vec{v}_s$ along $\hat{z}$ must be fixed at the walls (but not necessarily in the bulk). The appropriate thermodynamic potential[7] is then[8]

$$F=\int d^3r\{\tfrac{1}{2}\rho_s v_s^{\,2}-\tfrac{1}{2}\rho_0(\vec{l}\cdot\vec{v}_s)^2+c\vec{v}_s\cdot\nabla\times\vec{l}-c_0(\vec{l}\cdot\vec{v}_s)\vec{l}\cdot\nabla\times\vec{l}$$
$$+\tfrac{1}{2}K_s(\nabla\cdot\vec{l})^2+\tfrac{1}{2}K_t(\vec{l}\cdot\nabla\times\vec{l})^2+\tfrac{1}{2}K_b[\vec{l}\times(\nabla\times\vec{l})]^2\}\,. \qquad (2)$$

We can maintain the constraint on surface circulation by limiting the stability test to deviations from uniform flow that vanish at the surface—i.e. by setting a lower limit to the range of transverse wave vectors; the same device avoids the need to deal explicitly with the region of large y in which the equilibrium $\vec{l}$ turns toward the walls. The mathematical problem is therefore that of the second-order stability against arbitrary variations of the state given by $\vec{l}\equiv\hat{z}$, $\vec{v}_s\equiv w\hat{z}$ (with the understanding that any given geometry may exclude unstable modes of too long wavelength).

The stability test is straightforward if one recognizes that (1) requires a variation in $\vec{l}$ to be accompanied by a variation in $\vec{v}_s$ which must, to second order, be of the form

$$\delta\vec{v}_s = \tfrac{1}{2}(\lambda_x\nabla\lambda_y - \lambda_y\nabla\lambda_x) + \nabla\varphi, \tag{3}$$

where $\delta\vec{l} = \vec{\lambda} - \frac{1}{2}\lambda^2\hat{z}$, with $\lambda_z = 0$. Writing $\vec{\lambda}$ and φ in terms of their Fourier transforms, we find [after minimizing with respect to the $\varphi(\vec{k})$] that the second-order change in F is

$$\delta F = V^{-1}\sum_{\vec{k}}\{\tfrac{1}{2}K_s|\vec{q}\cdot\vec{\lambda}|^2 + \tfrac{1}{2}K_t|\vec{q}\times\vec{\lambda}|^2 + \tfrac{1}{2}K_b k_z^2|\vec{\lambda}|^2 + \tfrac{1}{2}\rho_0 w^2|\vec{\lambda}|^2 + (c_0 + \tfrac{1}{2}\rho_s{}^{\parallel})w\vec{k}\,\mathrm{Im}(\vec{\lambda}^*\times\vec{\lambda})$$
$$- (\rho_s q^2 + \rho_s{}^{\parallel}k_z^2)^{-1}\tfrac{1}{2}[\rho_0^2 w^2|\vec{q}\cdot\vec{\lambda}|^2 + c_0^2 k_z^2|\vec{q}\times\vec{\lambda}|^2 + \rho_0 c_0 w k_z q^2\,\mathrm{Im}(\vec{\lambda}^*\times\vec{\lambda})]\}, \tag{4}$$

where $\vec{q}$ is the projection of k in the x-y plane and $\rho_s{}^{\parallel} = \rho_s - \rho_0$.

When $\vec{k}$ is along the direction of flow $\hat{z}$ (more precisely, when q is of the order of the inverse transverse dimensions which we take to be large on the length scale set by $1/w$) then δF is minimum at

$$k_z{}^0 = [(c_0 + \tfrac{1}{2}\rho_s{}^{\parallel})/K_b]w. \tag{5}$$

The value of δF at $\vec{k} = k_z{}^0\hat{z}$ is positive provided

$$(c_0 + \tfrac{1}{2}\rho_s{}^{\parallel})^2/\rho_0 K_b < 1. \tag{6}$$

When the inequality (6) fails there is an instability at $\vec{k} = k_z{}^0\hat{z}$ associated with a circularly polarized transverse $\vec{\lambda}$, and a uniform drop in v_s given by the first term in (3) (the term in φ being negligible for small q).

For any reasonable flow the scale of spatial variation $1/w$ is large enough for the dipolar interaction effectively to lock the spin quantization axis $\vec{d}$ to the direction of $\vec{l}$. The parameters in the free energy (2) are then given in weak coupling near T_c by[8]

$$\rho_0 = c_0 = \rho_s{}^{\parallel} = \tfrac{1}{2}\rho_s, \quad K_t = K_s = K_b = \tfrac{5}{4}\rho_s. \tag{7}$$

These values give a critical ratio (6) of 0.9. It can be shown that with the parameters given by (7) δF is bounded below by its $q = 0$ value for any k_z. Therefore uniform superflow in the weak-coupling dipole-locked Ginzburg-Landau regime just passes the stability test.

Strong-coupling corrections to the parameters (7) have been roughly estimated to be a few percent,[9] and so it is conceivable that these might be destabilizing. Furthermore, as the temperature drops to zero, the weak-coupling values of K_b and ρ_s approach constants while ρ_0 vanishes according to a power law and c_0 diverges logarithmically.[8] Thus at low enough temperatures the critical ratio almost certainly exceeds unity. Whether this can happen at temperatures above the A-B transition is problematical, but the possibility exists, and the fact that there is nothing in principle to forbid it is quite intriguing.

When (6) fails to hold is there merely a slight relaxation to a nearby configuration, or does the instability herald the appearance of a catastrophic vortex texture which devours the supercurrent? Just past threshold some intricate analysis is required to answer this question. To indicate what may happen we examine, instead, the unrealistic but amusing case in which the parameters have the values

$$\rho_0 = c_0 = \rho_s{}^{\parallel} = \tfrac{1}{2}\rho_s, \quad K_b = 3K_t = 3K_s = \tfrac{3}{4}\rho_s, \tag{8}$$

leading to a critical ratio (6) of 1.5. The amusement lies in the fact that the weak-coupling parameters would have these values near T_c if the dipolar interaction did not lock $\vec{d}$ to $\vec{l}$.[8] Thus if uniform bulk superflow is in fact stable in ^{3}He-A, it owes its stability to the magnetic interaction between nuclei!

When the parameters are given by (8) *there is a continuous family of nonsingular configurations taking one from the original flow state to one with zero mass current, and such that the free energy decreases monotonically as the entire family of configurations is traversed.* Except near the surface we take the orbital part of the order parameters in the family (parametrized by τ) to be given by[10]

$$\vec{\phi}^{(1)} + i\vec{\phi}^{(2)} = e^{i(w-u)z}R(\hat{z}, -uz)R(\hat{y}, \tau)(\hat{x} + i\hat{y}), \quad 0 \le \tau \le \pi, \tag{9}$$

where $R(\hat{n}, \theta)$ is a rotation through θ about $\hat{n}$. Macroscopic periodicity along $\hat{z}$ (which stabilizes the superflow in ^{4}He) requires that w and u are integral multiples of $2\pi/L$, so that neither can vary with τ. These order parameters give

$$\vec{l} = \vec{\phi}^{(1)} \times \vec{\phi}^{(2)} = \hat{z}\cos\tau + \sin\tau(\hat{x}\cos uz - \hat{y}\sin uz),$$
$$\vec{v}_s = \phi_i{}^{(1)}\nabla\phi_i{}^{(2)} = [w - u(1 - l_z)]\hat{z}. \tag{10}$$

When τ is 0 or π, $\vec{l}$ has the uniform values $\hat{z}$ or $-\hat{z}$, and the mass current is proportional to $\vec{v}_s$ and given by $\rho_s{}^{\parallel}w\hat{z}$ or $\rho_s{}^{\parallel}(w - 2u)\hat{z}$.

Substituting these forms into (2), we find the free-energy density

$$f=\tfrac{1}{2}w^2\{\tfrac{1}{2}K_t b^2\sin^4\tau+\tfrac{1}{2}K_b b^2\cos^2\tau\sin^2\tau-c_0[1-(1-\cos\tau)b]\,b\cos\tau\sin^2\tau+\tfrac{1}{2}[\rho_s-\rho_0\cos^2\tau][1-(1-\cos\tau)b]^2\},\qquad(11)$$

where $b=u/w$. [Note that $\tau=0$ is a local minimum of f for any u provided the stability condition (6) holds; when (6) fails, the point $\tau=0$ ceases to be a local minimum at a wave vector u given by (5).] When the parameters are taken from (8), then f becomes

$$f=\tfrac{1}{4}\rho_s w^2[\tfrac{5}{4}b^2-2b+1+(b-b^2)\cos\tau-(\tfrac{1}{2}-b+\tfrac{1}{4}b^2)\cos^2\tau],\qquad(12)$$

which decreases monotonically as τ goes from 0 to π, provided $1-\sqrt{3}/3<b<\sqrt{3}-1$. The choice $b=\frac{1}{2}$ gives the desired family.

There is thus nothing in principle to prevent the spontaneous hydrodynamic collapse of a bulk supercurrent when the stability condition (6) fails. The theoretically determined free-energy coefficients for the physically relevant case of dipolar locking are sufficiently close to the threshold of this instability that one should be alert to the possibility of collapse.

We conclude with some remarks on the significance of such collapse.

(1) The form (9) must be altered in the vicinity of the walls to bring the $\vec{\phi}^{(i)}$ into the plane of the walls (which then insures that the surface circulation in $\vec{v}_s$ is fixed throughout the family). This alters the free energy by surface terms, and therefore has negligible effect on the energetics of collapse in a wide enough channel. The physics of the collapse, however, lies in such surface configurations. Lacking a boojum[5] to unlock the configuration at the walls, the evolution of τ from 0 to π must result, near the surface, in a layer of nonzero vorticity, to reduce v_s from its fixed surface value w to the value (10) in the bulk. In the case $b=\frac{1}{2}$ in which the bulk current is reduced precisely to zero it can be shown that the resulting surface configuration is an array of doubly quantized coreless vortices.[11] Instead of nucleating in the center and moving out to the walls (like conventional vortex rings in a cylinder) these vortices develop continuously out of nothing at opposite surfaces, reducing the supercurrent as they form.

(2) Nearer to threshold the free-energy coefficients can give a range of unstable values of b that excludes the value $b=\frac{1}{2}$ necessary for zero current in the final configuration. Collapse may well continue to be possible in such cases, but it will then entail a series of instabilities and the final pattern of vortex textures at the surface may be quite complex.

(3) We believe that this instability of uniform flow is closely related to that recently discovered by Hook and Hall.[12] They test the hydrodynamic stability of uniform $\vec{v}_n$ directed along $\hat{l}$ with zero mass flow. Aside, however, from a Galilean transformation and the imposition of a lower bound on k_z to accommodate their finite geometry, the mathematics of stability should be the same in both cases. In confirmation of this, we have investigated the Hook-Hall configuration with the hydrodynamic equations given by Ho[13] and find a stability condition identical to (6) (as one must if the hydrodynamics is consistent with the principle of nonnegative entropy production). We therefore believe that when present, the instability we have described would appear dynamically in the Hook-Hall configuration. We disagree with them, however, on the regime of parameters for which the instability exists.

We are indebted to John Hook, whose lecture at Erice on his remarkable hydrodynamic instability set us thinking about the equilibrium stability of uniform flow. We would also like to thank W. F. Brinkman and G. E. Volovik for many fascinating discussions of vortex textures at the recent Gordon Conference on Quantum Solids and Fluids. This work was supported in part by the National Science Foundation under Grant No. DMR 74-23494 and through the Materials Science Center of Cornell University, Technical Report No. 2898.

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[5]For mathematical details of boojum see N. D. Mermin, in *Quantum Fluids and Solids*, edited by S. B. Trickey, E. D. Adams, and J. W. Dufty (Plenum, New York, 1977), p. 3; see also P. W. Anderson and R. G.

Palmer, *ibid.*, p. 23.

[6]N. Webster, in *New International Dictionary* (Merriam, Springfield, Mass., 1934), 2nd ed., p. 308. See also p. 2379. (In this, as in many other matters, the views of the third edition should be spurned.)

[7]See, for example, V. Ambegaokar, in *Superconductivity*, edited by P. R. Wallace (Gordon and Breach, New York, 1969), p. 117.

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[9]J. Serene and D. Rainer, to be published.

[10]See Ref. 1 for a definition of the $\vec{\phi}^{(i)}$ and their relation to $\vec{v}_s$ and $\vec{l}$.

[11]T.-L. Ho, Ph.D. thesis, Cornell University, 1977 (unpublished).

[12]J. R. Hook and H. E. Hall, to be published. Note, through, that Hook and Hall claim to find instability in the dipole-locked regime, which is incompatible with our general stability analysis.

[13]T.-L. Ho, in *Quantum Fluids and Solids*, edited by S. B. Trickey, E. D. Adams, and J. W. Dufty (Plenum, New York, 1977), p. 97. These differ slightly from the equations used by Hook and Hall.

Magnetic Resonance in a Concentrated Spin-Glass: Experiment and a Phenomenological Model

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The microwave spectrum of amorphous sputtered films of $Gd_{0.37}Al_{0.63}$ exhibits a resonance which moves rapidly to lower fields as temperature is decreased below the paramagnetic Curie temperature at 30 K and which exhibits no anomaly at the spin-glass ordering temperature of 16 K. The shift of the resonance field is interpreted as an effective anisotropy field arising from local demagnetizing fields of the inhomogeneous magnetic system. The theory is also compared to earlier resonance results on CuMn.

In this Letter we report on the microwave resonance spectrum of amorphous GdAl and describe a semiquantitative model for the resonance line positions. The conclusions are of wider interest for the study of spin-glasses or other inhomogeneous magnetic systems because our theory indicates that the resonance line positions act as a sensitive probe of the spatial inhomogeneity of the system while standard magnetic measurements yield information primarily about the spatially averaged properties.

Magnetic measurements[1-3] on an amorphous sputtered film of $Gd_{0.37}Al_{0.63}$ have revealed a paramagnetic Curie temperature of 30 K, indicating a predominance of ferromagnetic exchange interactions. However at 16 K there appeared a sharp maximum in the dc low-field magnetic susceptibility, indicating a spin-glass ordering which presumably arose from the presence of random antiferromagnetic interactions along with the ferromagnetic interactions. Evidence of spatial magnetic inhomogeneity has come from specific-heat measurements,[4] which indicate partially ferromagnetic clusters with dimensions of order 35 Å, while small-angle x-ray scattering measurements[5] indicate structural inhomogeneities of undetermined nature but of comparable dimensions.

Here we describe the magnetic resonance spectrum of the same GdAl material,[6] obtained with a standard Varian electron spin resonance spectrometer at 9.13 GHz. We observed a broad resonance ($\Delta H \sim 1$ kOe) and measured the resonance fields $H_{\parallel}$ and $H_{\perp}$ as a function of temperature with dc fields applied parallel and perpendicular, respectively, to the film plane. At high temperature the resonance corresponds to $g = 2$, as expected for the Gd^{3+} ion. At lower temperatures the parallel and perpendicular resonances move apart because of a well-known demagnetizing-field effect, and below the paramagnetic Curie temperature both resonances plunge steeply to lower fields, as shown in Fig. 1. In this region the resonance line broadens rapidly and can no longer be measured accurately below about 12.5 K. Also shown in Fig. 1 are $M_{\perp}$ and $M_{\parallel}$, the average static magnetizations at the fields $H_{\perp}$ and $H_{\parallel}$, as interpolated from previously reported magnetic measurements.[1]

Qualitatively similar resonance results have been obtained in earlier studies of CuMn alloys by Owen and co-workers[7,8] and by Griffiths,[9] who observed resonance fields shifting downward with decreasing temperature near the magnetic ordering. To interpret the shift, Owen and co-workers[7,8] assumed that CuMn was a simple antiferromagnet giving an antiferromagnetic resonance at