sistance.

*This work was supported by the U. S. Energy Research and Development Administration under Contract No. E(11-1)-3073.

†Present address: Physics Department, Massachusetts Institute of Technology, Cambridge, Mass. 02139.

[1]J. L. Sperling and F. W. Perkins, Phys. Fluids 17, 1857 (1974).

[2]S. J. Bucksbaum, Phys. Fluids 3, 418 (1960).

[3]W. E. Drummond and M. N. Rosenbluth, Phys. Fluids 5, 1507 (1962).

[4]E. R. Ault and H. Ikezi, Phys. Fluids 13, 2874 (1970).

[5]M. Porkolab and R. P. H. Chang, Phys. Fluids 13, 2054 (1970); also D. Pesme, G. Laval, and R. Pellat, Phys. Rev. Lett. 31, 203 (1973).

[6]The finite-length effect becomes important near the ion cyclotron frequency.

[7]R. Limpaecher and K. R. Mackenzie, Rev. Sci. Instrum. 44, 726 (1973).

[8]M. Porkolab, Phys. Fluids 11, 834 (1968).

# Theory of Strongly Turbulent Two-Dimensional Convection of Low-Pressure Plasma*

R. N. Sudan and M. Keskinen
*Laboratory of Plasma Studies, Cornell University, Ithaca, New York 14853*
(Received 28 January 1977)

The "direct interaction approximation" of Kraichnan as modified by Kadomtsev is employed to develop a strong turbulence theory which predicts both nonlinear frequency broadening and a power law for the spectrum of a two-dimensional convecting plasma.

In this Letter we consider well-developed strong turbulence in a low-pressure weakly ionized plasma confined in a strong magnetic field $B\hat{x}$ subjected to both a gradient in density $(\partial n/\partial z)\hat{z}$ and an electric field $\vec{E}_0 = -(\partial\varphi_0/\partial z)\hat{z}$. The electrons and ions suffer collisions predominantly with the background neutrals such that $\Omega_e \gg \nu_e$ and $\Omega_i \lesssim \nu_i$, where $\Omega_{e,i}$ and $\nu_{e,i}$ are the respective cyclotron and collisional frequencies. The difference in $\Omega$ and $\nu$ between electrons and ions gives rise to a mean current density $J\hat{y}$ resulting from the $\vec{v}_0 = \vec{E}_0\times\vec{B}/B^2$ drift of the electrons. Simon[1] and Hoh[1] have independently shown that this configuration is unstable if $\nabla n\cdot\nabla(-e\varphi_0)>0$, in exact analogy to the gravitational instability.[2] The unstable fluctuations have the nature of growing waves propagating in the direction of electron drift, i.e., $\varphi\sim\varphi(z)\exp[i(k_y y-\omega t)]$.

We shall be concerned here with the nonlinear development of these fluctuations leading eventually to a two-dimensional turbulent state. The model developed below applies directly to the $E$-region ionospheric density irregularities driven by the equatorial electrojet.[3-5] This theory predicts (i) that the power spectrum of the density fluctuations $\langle|\delta n_{\vec{k}}/n_0|^2\rangle \equiv I_{\vec{k}}$ is proportional to $|\vec{k}|^{-n}$, where $n$ ranges at most between 3 and 4, (ii) that $I_{\vec{k}}$ is proportional to $|\vec{v}_0|^m$, and (iii) that the resonance broadening of the power spectrum in frequency, $\Gamma_{\vec{k}}$, is proportional to $k^{2-n/2}$. If we take into account that radar observations of the equatorial $E$-region indicate $m\simeq 2$, then this theory which relates $m$ to $n$ predicts $n\simeq\frac{16}{5}$, which agrees with such ionospheric data as are available and also with numerical simulations.

It appears to us that Eqs. (7) and (8) which express the mathematical content of this model could represent convective plasma motion in a broad class of experimental configurations including large toroidal machines for plasma confinement. Although the sources of the instabilities are individual to particular experiments the nonlinear interaction terms may be common to them.

Under the assumptions of quasineutrality $Zn_i \simeq n_e \equiv n$ and isothermality, the basic equations for the electron and ion $(Ze)$ fluids are

$$\partial n/\partial t+\nabla\cdot n\vec{v}_e=0, \tag{1}$$

$$ne(\vec{E}+\vec{v}_e\times\vec{B})+T_e\nabla n+nm_e\nu_e\vec{v}_e=0, \tag{2}$$

$$-Zn_ie(\vec{E}+\vec{v}_i\times\vec{B})+T_i\nabla n_i+n_im_i\nu_i\vec{v}_i=0, \tag{3}$$

$$\nabla\cdot(\vec{J}_i+\vec{J}_e)=0. \tag{4}$$

On linearizing these equations about the equilibrium $n_0=N_0(1+z/L)$, $d\varphi_0/dz=a/(z+L)$, unstable fluctuations $\varphi=\tilde{\varphi}\exp\{i[k_y y+k_z z-(\omega_{\vec{k}r}+i\gamma_{\vec{k}})t]\}$, with $\vec{k}\cdot\vec{B}=0$ and $k_yL\gg 1$, have frequency and growth rate given by[3]

$$\omega_{\vec{k}r}=\vec{k}\cdot\vec{v}_0/(1+\psi), \tag{5}$$

$$\gamma_{\vec{k}}=[\psi/(1+\psi)]\{(\Omega_e/\nu_e)[k_y^2v_0/k^2L(1+\psi)]-k^2C_s^2/\nu_i\}, \tag{6}$$

where $\psi = \nu_e \nu_i/\Omega_e \Omega_i$. These waves are nondispersive. Moreover as their amplitude increases it has been shown[4] that they do not steepen, but instead, they are unstable to a perturbation in a direction perpendicular to their propagation. This leads to further mode coupling in different directions and the final spectrum tends to isotropy in the $y$-$z$ plane. Because of the lack of dispersion in these waves it is clear that the weak turbulence theory of nonlinearly interacting linear modes is not applicable. A more satisfactory approach is that based on the "direct interaction approximation" of Kraichnan.[6] For our purpose we adopt Kadomtsev's development[7] of Kraichnan's postulates. Combining Eqs. (1)–(4) we obtain

$$\partial n/\partial t + (\mu\, \hat{x} \times \nabla\varphi)\cdot\nabla n = D\nabla^2 n, \tag{7}$$

$$(\beta\mu\hat{x}\times\nabla\varphi)\cdot\nabla n + \mu(\nabla\varphi\cdot\nabla n + n\nabla^2\varphi) = D'\nabla_n^{\;2}, \tag{8}$$

with

$$\beta = (\mu_{\mathrm{H}}^{\;e} - \mu_{\mathrm{H}}^{\;i})/(\mu_\perp^{\;e} + \mu_\perp^{\;i}),$$

$$D = (\mu_\perp^{\;i} D_\perp^{\;e} + \mu_\perp^{\;e} D_\perp^{\;i})/(\mu_\perp^{\;e} + \mu_\perp^{\;i}),$$

$$\mu = (\mu_\perp^{\;i}\mu_{\mathrm{H}}^{\;e} + \mu_\perp^{\;e}\mu_{\mathrm{H}}^{\;i})/(\mu_\perp^{\;e} + \mu_\perp^{\;i}),$$

$$D' = (\mu_\perp^{\;i}\mu_{\mathrm{H}}^{\;e} + \mu_\perp^{\;e}\mu_{\mathrm{H}}^{\;i})(D_\perp^{\;e} - D_\perp^{\;i})/(\mu_\perp^{\;e} + \mu_\perp^{\;i})^2.$$

Here $\mu_\perp$ and $\mu_{\mathrm{H}}$ are the Pedersen and Hall mobilities and $D_\perp$ is the diffusion transverse to the magnetic field.

Equations (7) and (8) can also be written in the Fourier representation as two coupled equations for $n_{\vec{k}\omega}$ and $\varphi_{\vec{k}\omega}$, where

$$f_{\vec{k}\omega} = (2\pi)^{-3}\int d^2x\, dt \int f(x,t) \exp[i(\vec{k}\circ\vec{x} - \omega t)].$$

To second order in $n_{\vec{k},\omega}$ we obtain, following standard techniques,

$$(\omega - \omega_{\vec{k}})n_{\vec{k},\omega} = \int d^2k'\, d\omega'\, V_{\vec{k},\vec{k}'} n_{\vec{k}-\vec{k}',\omega-\omega'} n_{\vec{k}',\omega'}, \tag{9}$$

where $\omega_{\vec{k}} = \omega_{\vec{k}r} + i\gamma_{\vec{k}}$ is given by (5) and (6) and

$$V_{\vec{k},\vec{k}'} = [\vec{k}'\cdot\hat{x}\times\vec{v}_0 - \beta(\vec{k}'\cdot\vec{v}_0 + i(k'C_s)^2/\nu_i](\vec{k}-\vec{k}')\times\hat{x}\cdot\vec{k}')/(\vec{k}')/(\vec{k}'\cdot\vec{k}'). \tag{9a}$$

One of the main consequences of strong nonlinear interaction is the self-damping of the modes $\Gamma_{\vec{k},\omega}$ which could far exceed the linear damping $\gamma_{\vec{k}}$. The density Fourier components $n_{\vec{k},\omega}$ are assumed to form a statistical ensemble with random phases. Following Kadomtsev's development[7] we arrive at the Fourier-transformed version of Kraichnan's equations, viz.,

$$|\omega - \omega_{\vec{k}} + \Gamma_{\vec{k},\omega}|^2 I_{\vec{k},\omega} = \tfrac{1}{2}\int d^2k'\, d\omega'\, |w_{\vec{k},\vec{k}'}|^2 I_{\vec{k}',\omega'} I_{\vec{k}-\vec{k}',\omega-\omega'}, \tag{10}$$

$$\Gamma_{\vec{k},\omega} = -\int d^2k'\, d\omega' \frac{w_{\vec{k},\vec{k}-\vec{k}'} w_{\vec{k}-\vec{k}',\vec{k}} I_{\vec{k}',\omega'}}{\omega - \omega' - \omega_{\vec{k}-\vec{k}'} + \Gamma_{\vec{k}-\vec{k}',\omega-\omega'}}, \tag{11}$$

where $\langle n_{\vec{k},\omega}{}^* n_{\vec{k}',\omega'}/n_0^{\,2}\rangle = I_{\vec{k},\omega}\delta(\vec{k}-\vec{k}')\delta(\omega-\omega')$ and $w_{\vec{k},\vec{k}'} = V_{\vec{k},\vec{k}'} + V_{\vec{k},\vec{k}-\vec{k}'}$. Since $\Omega_e \gg \nu_e$, $\Omega_i \ll \nu_i$, $\psi \ll 1$, and $\beta \simeq \nu_i/\Omega_i(1+\psi) \gg 1$, in the application we have in mind, we obtain, keeping terms proportional to $\beta^2$,

$$w_{\vec{k},\vec{k}-\vec{k}'} w_{\vec{k}-\vec{k}',\vec{k}} = -\beta^2(\hat{x}\times\vec{k}\cdot\vec{k}')^2\{[(\vec{k}\cdot\vec{v}_0)^2 - (\vec{k}'\cdot\vec{v}_0)(\vec{k}\cdot\vec{v}_0)]/k^2(\vec{k}-\vec{k}')^2$$
$$+ [(\vec{k}'\circ\vec{v}_0)(\vec{k}\circ\vec{v}_0) - (\vec{k}'\circ\vec{v}_0)^2]/k'^2(\vec{k}-\vec{k}')^2 - [k'^2(\vec{k}\circ\vec{v}_0) + k^2(\vec{k}'\cdot\vec{v}_0)](\vec{k}'\circ\vec{v}_0)/k^2k'^4\}. \tag{11a}$$

The self-consistent solution of Eqs. (10) and (11) for $I_{\vec{k},\omega}$ and $\Gamma_{\vec{k},\omega}$ is a formidable task. We shall first assume $I_{\vec{k},\omega}$ as given and solve for $i\Gamma_{\vec{k}} \equiv \Gamma_{\vec{k},\omega_{\vec{k}}}$. In this we are assisted by the numerical solutions of Eqs. (7) and (8) obtained by McDonald *et al.*[8] and Ferch and Sudan.[9] These calculations show that $\int d\omega\, I_{\vec{k},\omega} \propto k^{-n}$ with $n \simeq 3$–$4$.[10] We therefore approximate $I_{\vec{k},\omega}$ by

$$I_{\vec{k},\omega} = I_k g(\omega - \omega_{\vec{k}}),$$

where $g(\omega - \omega_{\vec{k}})$ is a resonant function which for simplicity we choose to be a Gaussian, $(2\pi\Gamma_k^{\,2})^{-1/2} \times \exp[-(\omega-\omega_{\vec{k}})^2/2\Gamma_k^{\,2}]$; $\Gamma_k = \int_0^{2\pi}\Gamma_{\vec{k}}\, d\theta/2\pi$ is the nonlinear width of the spectrum, $I_k \propto k^{-n}$, and $\vec{k} = (k,\theta)$. Substituting this expression in Eq. (11) we obtain ($\vec{k}'' = \vec{k} - \vec{k}'$)

$$\Gamma_k = -i\int_0^{2\pi}\frac{d\theta}{2\pi}\int d^2k'\, I_{k'} w_{\vec{k},\vec{k}''} w_{\vec{k}'',\vec{k}} \int_{-\infty}^{\infty}\frac{d\nu}{(2\pi\Gamma_{k'}^{\,2})^{1/2}}\frac{\exp[-(\nu^2/2\Gamma_{k'}^{\,2})]}{\nu - i\Gamma_{k''}}$$

$$= \frac{-i}{\sqrt{2}}\int_0^{2\pi}\frac{d\theta}{2\pi}\int d^2k'\, w_{\vec{k},\vec{k}''} w_{\vec{k}'',\vec{k}} \frac{I_{k'}}{|\Gamma_{k'}|} Z\left(\frac{i\Gamma_{k''}}{\sqrt{2}\,\Gamma_{k'}}\right). \tag{12}$$

In arriving at (12) we have replaced $\Gamma_{\vec{k}''}$ by its angular average $\Gamma_{k''}$ and furthermore taken $|\Gamma_{\vec{k}}| \gg |\gamma_{\vec{k}}|$; $Z$ is the plasma dispersion function. From (11a) and (12) we observe that $\Gamma_k$ is real since $Z(ix)$ is a purely positive imaginary quantity. Furthermore, since over most of the range of integration of $\vec{k}'$, $\Gamma_{k''}/\Gamma_{k'} \sim 1$, we obtain

$$\Gamma_k = 4.9 \int_{\alpha k}^{\infty} dk' k' (I_{k'}/|\Gamma_{k'}|) \int_0^{2\pi} (d\theta/2\pi) \int_0^{2\pi} d\theta' w_{\vec{k},\vec{k}''} w_{\vec{k}'',\vec{k}}. \tag{13}$$

Substituting for $w_{\vec{k},\vec{k}''} w_{\vec{k}'',\vec{k}}$ from (11) and performing the integrations over $\theta$ and $\theta'$ we obtain

$$\Gamma_k = -4.9(\pi/2)\beta^2 k^2 v_0^2 \int_{\alpha k}^{\infty} dk' k' I_{k'} F(k'/k)/|\Gamma_{k'}|, \tag{14}$$

where $F(k'/k)$ can be approximated by $\frac{3}{4}$ over the entire range of $k'/k$. We have furthermore adopted Kadomtsev's correction to the "direct interaction approximation" by taking the lower limit of the $k'$ integration to be proportional to $k$ with $\alpha \lesssim 1$. On solving for $\Gamma_k$ from (14) we obtain

$$|\Gamma_k| = 3.4 n^{-1/2} \alpha^{-n/4} \beta v_0 k^2 I_k^{1/2}. \tag{15}$$

It is clear from Eqs. (14) and (15) that $\Gamma_k$ is indeed a negative quantity and furthermore that $|\Gamma_k| \propto |v_0|$.

Figure 1(a) shows the power spectra from a numerical calculation[11] of Eqs. (7) and (8) after a steady turbulent state has been attained; $\Gamma_k/2\pi$ as measured from this curve is $\sim 2.1$ Hz, for $k = 2\pi/9$ m$^{-1}$. From the parameters of this calculation $\beta = (\Omega_e/\nu_e)\psi/(1+\psi) = 22.1$, $v_0 = 10^2$ m/sec, $\langle |\delta n/n_0|^2\rangle = 2\pi \int_{k_{\min}}^{\infty} dk\, k I_k = 2.0\times 10^{-3}$, $k_{\min} = 2\pi/128$, $k = 2\pi/9$, and $\theta = 45°$, we obtain $\Gamma_k/2\pi = 3.1$ Hz with $\alpha \sim \frac{1}{3}$ and $n = 3.2$ (this choice is discussed later). Figure 1 also shows the spectra for the $E$-region ionospheric irregularities obtained by radar[12] at Jicamarca, Peru, for ambient conditions approximately similar to those used in the numerical computations. The agreement between observed, numerical and analytically predicted frequency spread, as functions of $v_0$ and $k$, is reasonably good.

Turning now to Eq. (10) we obtain, after substituting for $I_{\vec{k}',\omega}$, and $I_{\vec{k}'',\omega''}$ and performing the $\omega'$ integration,

$$I_k = \int_0^{2\pi} \frac{d\theta}{2\pi} d\omega\, I_{\vec{k},\omega} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{2\pi} \int d^2k' \, |w_{\vec{k},\vec{k}'}|^2 I_{k'} I_{k''} \int_{-\infty}^{\infty} \frac{d\omega \exp[-(\omega-\omega_{\vec{k}})^2/2(\Gamma_{k'}^2+\Gamma_{k''}^2)]}{\sqrt{2\pi}(\Gamma_{k'}^2+\Gamma_{k''}^2)^{1/2}[(\omega-\omega_{\vec{k}})^2+\Gamma_k^2]}$$

$$= \frac{-i}{2\sqrt{2}} \int_{\alpha k}^{\infty} dk' k' \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^{2\pi} \frac{d\theta' \, |w_{\vec{k},\vec{k}'}|^2}{\Gamma_k(\Gamma_{k'}^2+\Gamma_{k''}^2)^{1/2}} I_{k'} I_{k''} Z\left(\frac{i\Gamma_k}{\sqrt{2}\,(\Gamma_{k'}^2+\Gamma_{k''}^2)^{1/2}}\right)$$

$$\sim \Gamma_k^{-1} \int_{\alpha k}^{\infty} dk' k' \Gamma_{k'}^{-1} \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^{2\pi} d\theta' \, |w_{k,k'}|^2 \left(1+\frac{\Gamma_{k''}^2}{\Gamma_{k'}^2}\right)^{-1/2} I_{k'} I_{k''} \sim \frac{\beta^2 k^2 v_0^2}{\Gamma_k} \int_{\alpha k}^{\infty} dk' k' I_{k'}^2 \Gamma_{k'}^{-1} G\left(\frac{k'}{k}\right). \tag{16}$$

In reaching (16) we have replaced $\Gamma_k$ in the resonant term of (10) by the angle-averaged quantity $\Gamma_k$; notice that $-iZ(ix)$ varies between 1.77 and 0.93 over the range of integration of $k'$. Now $G(k'/k)$ is only very weakly dependent on $k'/k$ and we replace it by its average value $\bar{G}$ which is of order unity. On substituting for $\Gamma_k$ from (15) we observe that $I_k \propto k^{-n}$ satisfies (16) thus establishing the self-consistency of our assumed solution.

Let the spectrum $I_k = Ik^{-n}$ extend from $k_{\min}$ to $k_{\max}$. Then in the steady state the total power generated in the unstable range $k_{\min}$ to $k_c$ must be balanced by the power dissipated between $k_c$ and $k_{\max}$ by the damped spectra. Thus

$$\int_{k_{\min}}^{k_{\max}} dk\, k \int_0^{2\pi} d\theta \gamma_{\vec{k}} I_k = 0. \tag{17}$$

From Eq. (17) we readily obtain

$$k_{\min}^{n-2} k_{\max}^{4-n} = k_c^2(4-n)/2(n-2), \tag{18}$$

where $k_c^2 = (\beta v_0/L\psi)\nu_i/C_s^2$ is the marginally unstable mode. Over the region of strong turbulence, viz. $k_{\min}$ to $k_{\max}$, the nonlinear frequency broadening $\Gamma_k \gg \gamma_{\vec{k}}$. However at large $k$ the linear damping $\gamma_{\vec{k}}$ must eventually dominate since the power spectrum drops off at large $k$. We therefore postulate that $k_{\max}$ is itself set by the condition where $\gamma_{\vec{k}}$ becomes comparable to $\Gamma_k$, i.e., by

$$\Gamma_{k_{\max}} = \gamma_{k_{\max}}. \tag{19}$$

From Eq. (15) and the expression for $\gamma_{\vec{k}}$ we obtain

$$k_{\max}^{n/2} = 3.4 n^{-1/2} \alpha^{-n/4} \frac{\Omega_e}{\nu_e} v_0 \frac{C_s^2}{\nu_i} I^{1/2}. \tag{19a}$$

The additional postulate (19) enables us to evaluate the absolute strength of the spectrum. Thus

$$\langle |\delta n/n_0|^2\rangle \equiv 2\pi\int_{k_{\rm min}}^{k_{\rm max}} dk\, k\langle |n_k/n_0|^2\rangle = 2\pi\int_{k_{\rm min}}^{k_{\rm max}} dk\, kIk^{-n}$$

$$= 0.6n\alpha^{n/2}\left(\frac{4-n}{(2n-4)(1+\psi)}\right)^{n/(4-n)}\left(\frac{\Omega_e}{\nu_e}\frac{v_0\nu_i L}{C_s^2}\right)^{(3n-8)/(4-n)}(k_{\rm min}L)^{(8-4n)/(4-n)}, \tag{20}$$

where we have utilized (18) and (19a) to eliminate $I$ and $k_{\rm max}$, in favor of $k_{\rm min}$. From Eq. (18) we observe that $2<n<4$. Furthermore radar observations at 3 m indicate that $\langle |n_k^+/n_0|^2\rangle \sim v_0^{\,m}$ with $m \simeq 2$, although occasional departures from this value are also recorded.[13] For this dependence $n \simeq \frac{16}{5}$ from Eq. (20). With this value of $n$, $L\sim 6$ km, $\alpha \sim \frac{1}{3}$, and adopting the values quoted earlier for the other parameters we obtain

$$\langle |\delta n/n_0|^2\rangle^{1/2} = 250(k_{\rm min}L)^{-3}v_0/100,$$

where $L$ is in meters and $v_0$ is in meters/second. It is unreasonable to expect that the spectrum will be isotropic and retain the power-law dependence for $k_{\rm min}L \sim 1$. Indeed the matrix coefficients $V_{k,k'}$ have been computed on the assumption that $kL \gg 1$. For $\lambda_{\rm max} \sim L/4$ or $k_{\rm min}L \sim 25$, we obtain $\langle |\delta n/n_0|^2\rangle^{1/2} \sim 0.02$ which is not inconsistent with such observations as are presently available.

In conclusion the theory developed here predicts $I_k \propto k^{-n}$ with $n \simeq 3.2$ consistent with experimental observations of $I_k \propto v_0^{\,2}$. In addition it predicts the nonlinear broadening of the spectrum $\Gamma_k \propto k^{0.4}$ and finally it gives a reasonable estimate of the absolute magnitude of the density fluctuations.

We are grateful to D. T. Farley for many useful discussions.

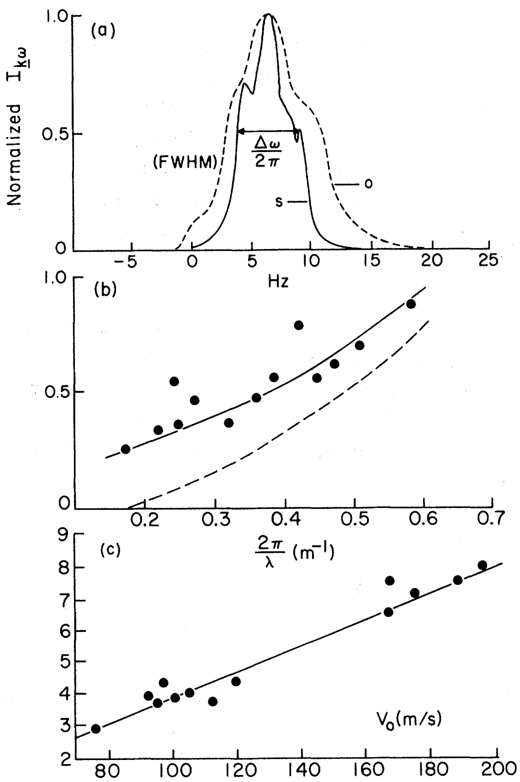


FIG. 1. Power spectrum $I_{\vec{k},\omega}$ vs frequency in Hz; curve $s$ for $k=2\pi/9$ m$^{-1}$ is obtained from numerical simulation of Eqs. (7) and (8), Ref. 11; curve $o$ is from radar backscatter observations at 16 MHz ($\lambda = 9.4$ m), Jicamarca, Peru, Ref. 12; $\gamma_k/2\pi = 0.9$ Hz for parameters of curve $s$; $\Delta\omega/2\pi$ is the full width at half-maximum of the power spectrum. (b) Dots show $(\Delta\omega/2\pi)/(8\ln 2)^{1/2}$ vs $k$ from simulation studies; dotted curve gives $\gamma_k/2\pi$ vs $k$ and solid line is $(\Gamma_k^{\,2}+\gamma_k^{\,2})^{1/2}/2\pi$ with numerical factor adjusted for best fit. At large wave numbers $\gamma_k$ is comparable to $\Gamma_k$ in simulation studies. (c) Dots show $(\Delta\omega/2\pi)/(8\ln 2)^{1/2}$ vs electrojet drift velocity from radar observations; solid line is $\Gamma_k/2\pi$ with adjusted numerical coefficient. Notice the excellent agreement with the analytical result $\Gamma_k \propto v_0$.

*Work supported by National Science Foundation Grant No. DE575-02797 and by U. S. Office of Naval Research Contract No. N00014-76-C-0359.

[1]A. Simon, Phys. Fluids 6, 382 (1963); F. C. Hoh, Phys. Fluids 6, 1184 (1963).

[2]M. N. Rosenbluth and C. Longmire, Ann. Phys. (N.Y.) 1, 120 (1957).

[3]A. Rogister and N. D'Angelo, J. Geophys. Res. 75, 3879 (1970).

[4]R. N. Sudan, J. Akimrimisi, and D. T. Farley, J. Geophys. Res. 78, 240 (1973).

[5]B. B. Balsley and D. T. Farley, J. Geophys. Res. 78, 7471 (1973).

[6]R. H. Kraichnan, J. Fluid Mech. 5, 497 (1959).

[7]B. B. Kadomtsev, *Plasma Turbulence* (Academic, New York, 1965), Chap. III.

[8]B. E. McDonald, T. P. Coffey, S. L. Ossakow, and R. N. Sudan, J. Geophys. Res. 79, 2551 (1974).

[9]R. L. Ferch and R. N. Sudan, Laboratory of Plasma Studies, Cornell University, Report No. 198, 1976 (unpublished).

[10]From dimensional analysis $n \geq 3$ for $m=2$; E. Ott and

D. T. Farley, J. Geophys. Res. 79, 2469 (1974).
[11]M. Keskinen, R. L. Ferch, and R. N. Sudan, Bull. Am. Phys. Soc. 21, 1115 (1976).
[12]B. B. Balsley and D. T. Farley, J. Geophys. Res. 76, 8341 (1971).
[13]B. B. Balsley, J. Geophys. Res. 74, 2333 (1969).

# Overlap of Bounce Resonances and the Motion of Ions in a Trapped-Ion Mode

Gary R. Smith
*Lawrence Berkeley Laboratory, University of California, Berkeley, California 94720*
(Received 3 March 1977)

Motion near a separatrix (the boundary between closed and open orbits) is studied both theoretically and numerically for parameters appropriate to the dissipative trapped-ion mode. I observe disappearance of an invariant in a stochastic layer surrounding the separatrix, as a result of overlapping bounce resonances. The fraction of ions lying within the stochastic layer is large even for a mode of relatively small amplitude.

Dissipative trapped-particle instabilities[1] are thought to cause anomalous transport in tokamaks. The transport rates are determined by the amplitude of the fluctuating electric field of the nonlinearly saturated instability. It is therefore important to consider nonlinear processes which might lead to saturation at relatively low levels.

In this Letter I report studies of a nonlinear process not previously considered, as far as I know, for the trapped-ion mode. Some stabilizing effect of this process have already been studied by us, but estimates of saturation levels are reserved for publication elsewhere, because details of trapped-particle instability theory are required.

The equations of motion which we study describe other physical situations of current interest. The equations were first used[2] to study the one-dimensional motion of a particle in two electrostatic waves with different amplitudes and phase velocities. This configuration is of interest[3] in the theory of radio-frequency heating. The equations were used[4] for a tokamak to estimate the fraction of a magnetic island (due to a tearing mode) in which field lines would be braided. In the general theory of stochasticity,[5] study of motion near a separatrix is recognized as being of fundamental importance.

We study the guiding-center motion of an ion in a magnetic field whose amplitude varies sinusoidally, the usual model[6] for a tokamak of large aspect ratio and circular cross section. The inclusion of guiding-center drifts off a field line would not change the physics of the process being studied, so we ignore those drifts. The calculations are thus in the spirit of local theory, in which radial excursions are assumed to be sufficiently small. We ignore collisions in order to isolate the effects, including pitch-angle scattering, due to the electric field of the mode.

A trapped-ion mode causes perturbations in the motion of the ion. We assume that a single mode is present, with toroidal and poloidal mode numbers $l$ and $m$, respectively. As the ion moves along a field line, it feels the potential

$$\Phi(\Theta, t) = -\Phi_0 \cos[(m - lq)\Theta - \omega t + \eta], \tag{1}$$

where $\Theta$ is the poloidal angle, $q$ the safety factor on the magnetic surface on which the ion moves, $\omega$ the mode frequency, and $\eta$ a phase angle related to the ion's initial toroidal angle.

The equations of motion are derivable from the Hamiltonian

$$H(\Theta, p, t) = H_0(\Theta, p) + e\Phi(\Theta, t),$$
$$H_0 = (p/qR_0)^2/2M - \mu\Delta B \cos\Theta, \tag{2}$$

where $p$ is the momentum conjugate to $\Theta$, $R_0$ the major radius of the tokamak, $e$ and $M$ the ion charge and mass, $\Delta B$ the modulation amplitude of the magnetic field, and $\mu$ the magnetic moment. Since $\mu$ is conserved during the motion, it plays the role of a parameter here.

A more convenient form of Hamiltonian (2) for analytic work uses action and angle variables. We express the unperturbed Hamiltonian $H_0$ in terms of the longitudinal action

$$J(H_0) = (2\pi)^{-1}\oint p\, d\Theta,$$

which, except for constant factors, is $\oint v_\parallel ds$, in standard notation. The explicit expressions[2,7] for $J$ and the canonically conjugate angle variable $\varphi$ involve elliptic integrals and $\kappa$, defined by $2\kappa^2 \equiv (1 + H_0/\mu\Delta B)$. The variables $J$ and $\varphi$ are defined in such a way that an ion which changes from trapped to circulating does not change its $J$ or $\varphi$