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Two-Dimensional Ising Model as an Exactly Solvable Relativistic Quantum Field Theory: Explicit Formulas for n-Point Functions*

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For both $T > T_c$ and $T < T_c$, we report exact expressions for the n-spin correlation function of the two-dimensional Ising model suitable for studying large separations between spins. In particular, the scaling limit of these correlation functions can be shown to exist and yields n-point Schwinger functions of some relativistic quantum field theories.

The last decade has seen the widespread recognition of the close relation between relativistic quantum field theory, with or without spontaneous symmetry breaking, and the statistical mechanics of ferromagnets on a lattice near the critical temperature T_c.[1,2] As $T \to T_c$, since the correlation length of a ferromagnet diverges as $|1 - T/T_c|^{-\nu}$, it is natural to scale all distances R according to

$$r = |1 - T/T_c|^{\nu} R. \tag{1}$$

In the so-called scaling limit, where $R_i \to \infty$, $T \to T_c$ with all r_i fixed, a suitable choice of β will permit the existence of

$$S(r_1, r_2, \ldots, r_n) = \lim |1 - T/T_c|^{-n\beta} \langle \sigma_{R_1} \sigma_{R_2} \cdots \sigma_{R_n} \rangle. \tag{2}$$

This scaled n-point correlation function of the ferromagnet is, up to a possible multiplicative constant, the n-point Schwinger function of the quantum field theory.[3]

The statistical model on which we have the most information is the two-dimensional Ising model.[4] In this note, we report the exact expressions for $\langle \sigma_{R1} \sigma_{R2} \cdots \sigma_{R_n} \rangle$, both above and below T_c.

Some time ago Montroll, Potts, and Ward[5] derived a representation of the multispin correlation functions of the two-dimensional Ising model in terms of determinants. These determinants are of small size when the spins can be grouped into a set of pairs such that the members of each pair are close together. However, if all spins are widely separated the size of the determinant grows with the separation and the behavior of the correlation functions in the scaling limit is no longer manifest.

The process of converting the determinants of Montroll, Potts, and Ward into a form useful for studying widely separated spins was initiated, over a decade ago, for the two-point function $\langle \sigma_{M_1 N_1} \sigma_{M_2 N_2} \rangle$.[6] However, these results deal only with the case of large r, i.e.,

$$|1 - T/T_c| [(M_1 - M_2)^2 + (N_1 - N_2)^2]^{1/2} \gg 1.$$

The complete expression for this two-point function, useful in particular in the scaling limit, has been obtained only recently.[7]

The method followed in Ref. 7 to deal with the two-point function begins with a particular choice, already used in Ref. 6, of the determinant of Montroll, Potts, and Ward. This particular choice has no sensible generalization to the n-point function with $n>2$. It is a recent realization that a different choice leads to a natural generalization to all n. The results[8] are remarkably compact and the derivation is no more complicated than for for the case $n=2$. The derivation will be published at a later date. The results are as follows:

(a) For $T<T_c$,

$$\langle\sigma_{M_1N_1}\sigma_{M_2N_2}\cdots\sigma_{M_nN_n}\rangle = \mathfrak{M}^n \exp F_n, \tag{3}$$

where

$$\mathfrak{M} = \left\{1-\left[\sinh\left(\frac{2E_1}{kT}\right)\sinh\left(\frac{2E_2}{kT}\right)\right]^{-2}\right\}^{1/8}, \tag{4}$$

$$F_n = \sum_{k=2}^{\infty} F_n^{(k)}, \tag{5}$$

with

$$F_n^{(k)} = -\frac{1}{2k}\left[2z_2(1-z_1^2)\right]^k \int_{-\pi}^{\pi}\frac{d\varphi_1}{2\pi}\cdots\frac{d\varphi_{2k}}{2\pi} \times\prod_{l=1}^{k}\frac{1}{\Delta(\varphi_{2l-1},\varphi_{2l})}\frac{\sin\frac{1}{2}(\varphi_{2l-1}+\varphi_{2l+1})}{\sin\frac{1}{2}(\varphi_{2l}-\varphi_{2l+2}+i\epsilon)}\,\mathrm{Tr}[A(1,2)A(3,4)\cdots A(2k-1,2k)], \tag{6}$$

where $A(2l-1,2l)$ is an $n\times n$ matrix with elements

$$A(2l-1,2l)|_{jj}=0$$

and

$$A(2l-1,2l)|_{jk} = \mathrm{sgn}(M_{jk})\exp(-iM_{jk}\varphi_{2l-1}-iN_{jk}\varphi_{2l}), \tag{7}$$

$$\varphi_{2k+1}\equiv\varphi_1,\quad \varphi_{2k+2}\equiv\varphi_2,\quad M_{\alpha\beta}=M_\alpha-M_\beta,\quad N_{\alpha\beta}=N_\alpha-N_\beta, \tag{8}$$

$$\Delta(\varphi_1,\varphi_2) = (1+z_1^2)(1+z_2^2)-2z_2(1-z_1^2)\cos\varphi_1-2z_1(1-z_2^2)\cos\varphi_2, \tag{9}$$

$$z_1=\tanh(E_1/kT),\quad z_2=\tanh(E_2/kT), \tag{10}$$

with E_1 (E_2) the horizontal (vertical) interaction energy; k is Boltzmann's constant; $\mathrm{sgn}x=+1$ if $x>0$, -1 if $x<0$, and either ±1 if $x=0$; and the limit $\epsilon\to0^+$ is understood.

(b) For $T>T_c$ (with n even since by symmetry the correlation function vanishes if n is odd),

$$\langle\sigma_{M_1N_1}\sigma_{M_2N_2}\cdots\sigma_{M_nN_n}\rangle = \mathfrak{M}_>^n G_n e^{F_n}, \tag{11}$$

where F_n is still given by (5) and (6);

$$\mathfrak{M}_> = \left\{\left[\sinh\left(\frac{2E_1}{kT}\right)\sinh\left(\frac{2E_2}{kT}\right)\right]^{-2}-1\right\}^{1/8}, \tag{12}$$

$$G_n = |\det G_{(n)ij}|^{1/2},\quad i=1,2,\ldots,n;\ j=1,2,\ldots,n, \tag{13}$$

and

$$G_{(n)ij} = \sum_{k=1}^{\infty} G_{(n)ij}^{(k)}, \tag{14}$$

with

$$G_{(n)ii}^{(k)}=0$$

and

$$G_{(n)ij}^{(k)} = -G_{(n)ji}^{(k)} = [2z_2(1-z_1^2)]^{k-1/2}[2z_1(1-z_2^2)]^{1/2}\int_{-\pi}^{\pi}\frac{d\varphi_1}{2\pi}\cdots\frac{d\varphi_{2k}}{2\pi}\prod_{l=1}^{k}\frac{1}{\Delta(\varphi_{2l-1},\varphi_{2l})} \times\prod_{l=1}^{k-1}\left(\frac{\sin\frac{1}{2}(\varphi_{2l-1}+\varphi_{2l+1})}{\sin\frac{1}{2}(\varphi_{2l}-\varphi_{2l+2}+i\epsilon)}\right)\exp[i\tfrac{1}{2}(\varphi_{2k-1}-\varphi_1)]\exp[i\tfrac{1}{2}(\varphi_{2k}-\varphi_2)] \times\{A(1,2)A(3,4)\cdots A(2k-1,2k)\}_{ij}, \tag{15}$$

where $\{\Theta\}_{ij}$ is the ij element of the matrix Θ.

To take the scaling limit we define

$$m_{\alpha\beta}=M_{\alpha\beta}|z_1z_2+z_1+z_2-1|[z_2(1-z_1^2)]^{-1/2}, \tag{16a}$$

$$n_{\alpha\beta}=N_{\alpha\beta}|z_1z_2+z_1+z_2-1|[z_1(1-z_2^2)]^{-1/2}, \tag{16b}$$

and let $M_{\alpha\beta}\to\infty$, $N_{\alpha\beta}\to\infty$, $z_1z_2+z_1+z_2-1\to 0$ $(T\to T_c)$ with $m_{\alpha\beta}$ and $n_{\alpha\beta}$ fixed to obtain, (a) for $T<T_c$,

$$\lim \mathfrak{M}^{-n}\langle\sigma_{M_1N_1}\sigma_{M_2N_2}\cdots\sigma_{M_nN_n}\rangle=\exp f_n \tag{17}$$

with

$$f_n=\sum_{k=2}^{\infty}f_n^{(k)}, \tag{18}$$

$$f_n^{(k)}=-\frac{1}{2k}(2\pi^2)^{-k}\int_{-\infty}^{\infty}dx_1\cdots dx_k dy_1\cdots dy_k\prod_{l=1}^{k}(1+x_l^2+y_l^2)^{-1}\frac{y_l+y_{l+1}}{x_l-x_{l+1}+i\epsilon}\mathrm{Tr}[a(1)a(2)\cdots a(k)], \tag{19}$$

where $a(l)$ is an $n\times n$ matrix with elements

$$a(l)|_{jj}=0,$$

$$a(l)|_{jk}=\mathrm{sgn}(m_{jk})\exp(-im_{jk}y_l-in_{jk}x_l); \tag{20}$$

(b) for $T>T_c$,

$$\lim\mathfrak{M}_>^{-n}\langle\sigma_{M_1N_1}\sigma_{M_2N_2}\cdots\sigma_{M_nN_n}\rangle=g_n\exp f_n, \tag{21}$$

where f_n is still given by (18) and (19) and

$$g_n=|\det g_{(n)ij}|^{1/2}, \tag{22}$$

$$g_{(n)ij}=\sum_{k=1}^{\infty}g_{(n)ij}^{(k)} \tag{23}$$

with $g_{(n)ii}^{(k)}=0$ and

$$g_{(n)ij}^{(k)}=-g_{(n)ji}^{(k)}=(2\pi^2)^{-k}\int_{-\infty}^{\infty}dx_1\cdots dx_k dy_1\cdots dy_k\prod_{l=1}^{k}(1+x_l^2+y_l^2)^{-1}$$

$$\times\prod_{l=1}^{k-1}\frac{y_l+y_{l+1}}{x_l-x_{l+1}+i\epsilon}\{a(1)a(2)\cdots a(k)\}_{ij}. \tag{24}$$

The formulas (3) and (11) possess many properties which are not immediately obvious. In particular, because of the signature factors and the $+i\epsilon$ prescription, many properties which are true of (3) and (11) are not valid term by term in the expressions (6) and (15). We have explicitly verified invariances under (1) $M_\alpha\to -M_\alpha$, (2) $N_\alpha\to -N_\alpha$, and (3) $M_\alpha\leftrightarrow N_\alpha$ and $E_1\leftrightarrow E_2$, as well as rotational invariance in the scaling limit. The connection between representations (3) and (11) of the n-point function and the T near T_c expansion of previous authors[9] will be studied elsewhere.

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Critical Behavior of Random Resistor Networks*

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We present numerical data and scaling theories for the critical behavior of random resistor networks near the percolation threshold. We determine the critical exponents of a suitably defined resistance correlation function by a Padé analysis of low-concentration expansions as a function of dimensionality. We verify that $d=6$ is the critical dimensionality for the onset of mean-field behavior. We use the coherent-potential approximation to construct a mean-field scaling function for the critical region.

In this Letter we report some new ideas concerning the properties of random resistor networks near the percolation threshold.[1] The model we treat is that of an electrical network on a d-dimensional hypercubic lattice of L^d sites with conductances σ_{ij} connecting nearest neighboring pairs of lattice sites i and j. Each σ_{ij} is an independent random variable assuming the values $\sigma_<$ or $\sigma_>$ with respective probabilities $1-p$ and p. The macroscopic conductivity, Σ, is then defined to be the configurational average of σL^{2-d}, where $\sigma \equiv I/V$, where I is the current when the potential difference V is applied between two opposite $(d-1)$-dimensional faces of the hypercube. We may define clusters as being groups of sites which are connected with respect to the conductances $\sigma_>$. The statistics of cluster size and the associated pair connectedness correlation length, $\xi(p)$, were shown[2] to be related to the thermodynamics of the s-state Potts model in the limit $s \to 1$, if the identification $p=1-e^{-J/kT}$ is made, where J is the coupling constant for nearest-neighbor interaction in the Potts model. This relation indicates that the usual exponent description for phase transitions can be applied to the percolation threshold and that the various scaling relations and universality predictions can be expected to hold as well. It was later shown[3,4] that for $d>d_c=6$, mean-field theory gives correct values for cluster statistics near the percolation threshold: $\alpha=-1$, $\beta=1$, $\gamma=1$, and $\nu=\frac{1}{2}$. In view of scaling arguments which relate the resistor network and percolation problems, de Gennes[5] has suggested that $d_c=6$. Here we present numerical evidence which confirms that this suggestion is correct. We also discuss several new scaling relations.

A way to determine d_c without using the renormalization group (RG) is to analyze the high-tem-