Unified Geometric Theory of Gravity and Supergravity*

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A unified geometric formulation of gravitation and supergravity is presented. The action for these theories is constructed out of the components of the curvature tensor for bundle spaces with four-dimensional Lorentz base manifold and structure groups Sp(4) for gravity and OSp(1,4) for supergravity. The requirement of invariance under reflections, local Lorentz transformations, and general coordinate transformations uniquely determines the action and ensures the existence of local supersymmetry in supergravity.

A theory has recently been proposed in which the gravitational field and a massless spin-$\frac{3}{2}$ Majorana field are coupled in a supersymmetric way.[1,2] Other supersymmetric theories have also been obtained incorporating additional fields, including spin-$\frac{1}{2}$ and spin-1 fields.[3] These theories have all been constructed from some trial Lagrangian by adding to it successive terms so as to make the total Lagrangian supersymmetric. In spite of its success, we feel that this procedure is not completely satisfactory insofar as it does not explicitly exhibit the underlying gauge-group structure of the new symmetries.[4]

In this Letter, we present a unified formulation of Einstein's theory of gravitation and the original supersymmetric theory, which couples the gravitational field to a real massless Rarita-Schwinger field, hereafter referred to as supergravity. In this formulation all the fields $h_\mu{}^A$ are treated as gauge potentials, belonging to the adjoint representation of a gauge group or supergroup G, which has the Lorentz group SL(2C) as a subgroup.

The Lagrangian is constructed exclusively in terms of the curvature tensor $R_{\mu\nu}{}^A$, which transforms covariantly under G. It is required that the action be invariant under general coordinate transformations and under local Lorentz transformations [gauge transformations in the subgroup SL(2C) of G]. It is found that the most general Lagrangian so constructed, taking for G the de Sitter covering group Sp(4), is equivalent to Einstein's Lagrangian with the addition of the cosmological term. If one performs a Wigner-Inonü contraction of Sp(4) down to the Poincaré group, the cosmological term drops out.

Similarly, the Lagrangian constructed with gauge fields associated with the superalgebra OSp(1,4) is a generalization of the theory of Ref. 1 with the addition of cosmological terms. Again, after contraction it reduces to that theory.

The origin of the supersymmetry can be directly traced to the fact that in the absence of the vierbein potentials $h_\mu{}^i$ associated with translations, the action is invariant under the supergroup OSp(1,2C)—in fact, it becomes a topological invariant associated with this group. In this Letter, we shall outline the general formalism and give the main results of its application to the theory of gravitation and supergravity. A more complete and detailed description will be given in a subsequent publication.[5]

Consider a continuous group or supergroup G and let H be a maximal Lie subgroup of G. The Lie (super-) algebra L of G can be decomposed into the direct sum $L_0 \oplus L_1$, where L_0 is the Lie algebra of H and L_1 is homeomorphic to the quotient space G/H. Let $\{X_A\}=\{X_{A^0},X_{A^1}\}$ be a basis for L. We have the following product in L:

$$[X_A,X_B\} = -(-1)^{\sigma_A\sigma_B}[X_B,X_A\} = f_{BA}{}^C X_C, \qquad (1)$$

where the signature σ_A is defined by $\sigma_A=1$ if the index A belongs to a component of L_1 and G is a supergroup, $\sigma_A=0$ otherwise.[6] The structure constants $f_{BA}{}^C$ are easily identifiable for the groups that we shall be considering; and we need not use their explicit forms in terms of the Minkowski metric and γ matrices. However, we would like to emphasize that, following current practice in gauge theories, we incorporate all the coupling parameters in the structure constants.

The adjoint representation of G is a set of vector fields $\{h^A\}$ with the following transformation under L:

$$(X_A h^B) = f_{AC}{}^B h^C. \qquad (2)$$

The Killing metric of L is the constant tensor,[7]

$$g_{AB} = (-1)^{\sigma_A\sigma_B} g_{BA} = \sum_{C,D} -(-1)^{\sigma_C} f_{AC}{}^D F_{BD}{}^C. \qquad (3)$$

It satisfies the identity

$$f_{AC}{}^D g_{DB} - f_{BA}{}^D g_{CD} = 0. \qquad (4)$$

We construct a principal bundle $P(G, M_4)$ with structure group G and a four-dimensional real base manifold M_4. Let $\{x^\mu\}$ be a coordinate system in M_4. Introduce a connection in P by defining the covariant derivatives

$$D_\mu = \partial_\mu + h_\mu{}^A X_A, \tag{5}$$

where $h_\mu{}^A$ belongs to the adjoint representation of G, and transforms as a vector (with respect to the index μ) under general coordinate transformations. Then we have

$$[D_\mu, X_A] = 0. \tag{6}$$

The $h_\mu{}^A$'s are gauge potentials (or connection coefficients). Under a local infinitesimal gauge transformation generated by $\epsilon^A X_A$ we have

$$\delta_\epsilon D_\mu = [D_\mu, \epsilon^A X_A] = (D_\mu \epsilon^A) X_A, \tag{7}$$

so that

$$\delta_\epsilon h_\mu{}^A = D_\mu \epsilon^A = \partial_\mu \epsilon^A + f_{CB}{}^A h_\mu{}^C \epsilon^B. \tag{8}$$

The curvature $\mathcal{R}$ is a horizontal two-form with values in the Lie algebra of G, whose components are given by

$$\mathcal{R}(D_\mu, D_\nu) = -[D_\mu, D_\nu] = R_{\mu\nu}{}^A X_A, \tag{9}$$

where

$$R_{\mu\nu}{}^A = \partial_\nu h_\mu{}^A - \partial_\mu h_\nu{}^A + h_\nu{}^B h_\mu{}^C f_{BC}{}^A. \tag{10}$$

They transform covariantly $\delta_\epsilon R_{\mu\nu}{}^A = f_{CB}{}^A R_{\mu\nu}{}^C \epsilon^B = -f_{BC}{}^A \epsilon^B R_{\mu\nu}{}^C$. The Jacobi identities for the operators D_μ imply the following Bianchi identities

$$D_{(\lambda} R_{\mu\nu)}{}^A = 0, \tag{11}$$

where $(\lambda\,\mu\nu)$ stands for cyclic permutations of the indices. The most general integral over a cross section of the bundle, depending on the fields $h_\mu{}^A$ *only* through the components of $\mathcal{R}$ (polynomial in these fields) and invariant under general coordinate transformations, is of the form

$$I(Q) = \int R_{\mu\nu}{}^A R_{\lambda\sigma}{}^B Q_{AB} \epsilon^{\mu\nu\lambda\sigma} d^4x \tag{12}$$

or, in a coordinate independent way,

$$I(Q) = \int \mathcal{R}^A \wedge \mathcal{R}^B Q_{AB}, \tag{13}$$

where Q_{AB} are constants, antisymmetric if G is a supergroup and (A, B) belongs to L_1, but symmetric otherwise. The variation of I corresponding to infinitesimal transformations of the h_μ's given by (8) (local gauge transformations) is

$$\delta_\epsilon I = -\int 2\epsilon^C R_{\mu\nu}{}^D R_{\lambda\sigma}{}^B f_{CD}{}^A Q_{AB} \epsilon^{\mu\nu\lambda\sigma} d^4x. \tag{14}$$

The variation of I for arbitrary variations of the h's is

$$\delta I = \int 4\delta h_\mu{}^A h_\nu{}^B R_{\lambda\sigma}{}^C (f_{BC}{}^D Q_{AD} - f_{AB}{}^D Q_{DC}) \epsilon^{\mu\nu\lambda\sigma} d^4x. \tag{15}$$

In deriving this result, we make use of the Bianchi identities (11). It follows from (4) that for $Q_{AB} = g_{AB}$, or more generally for Q_{AB} an invariant tensor [which satisfies an equation of the form (4)], I is a topological invariant.

One can decompose $I(Q)$ into elements of irreducible representations of G. Such a decomposition which is relevant for the determination of the symmetries of I will be discussed in another paper.

We shall now show how the theories of gravitation and supergravity can be formulated within this framework in a unified manner. We require that these theories be invariant under local Lorentz transformations so that G must contain the Lorentz group as a subgroup. Take $G = \mathrm{Sp}(4)$ for the theory of gravitation and $G^* = \mathrm{OSp}(1, 4)$ for supergravity. The gauge potentials belonging to the adjoint representation are $\{h_\mu{}^a, h_\mu{}^i\}$ and $\{h_\mu{}^a, h_\mu{}^\alpha, h_\mu{}^i\}$, respectively. The index $a = [ij]$ (a pair of antisymmetric indices) is associated with Lorentz transformations X_a; the index i is a vector index associated with the generator X_i and α is a spinor index associated with the supersymmetry generator X_α. Then $h_\mu{}^a$, $h_\mu{}^i$, amd $h_\mu{}^\alpha$ are, respectively, the gauge potential for the Lorentz group (connection coefficients), the vierbein and the spin-$\frac{3}{2}$ Majorana field. The corresponding gauge fields are the components of the curvature tensor $R_{\mu\nu}{}^A$ defined for each of these two groups by (10).

The general form of the action integral I constructed with $R_{\mu\nu}{}^A$, invariant under local Lorentz transformations and even under reflections, is obtained with the following choice for Q_{AB}:

(i) $Q_{AB} = \epsilon_{ab}$ for the group $G = \mathrm{Sp}(4)$,

(ii) $Q_{AB} = \{\epsilon_{ab}, \chi(\gamma_5)_{\alpha\beta}\}$ for the supergroup $G^* = \mathrm{OSp}(1, 4)$

where $\epsilon_{ab} = \epsilon_{ijkl}$ is the Levi-Civitá symbol, $(\gamma_5)_{\alpha\beta} = C_{\alpha\gamma}(\gamma_5)_\beta{}^\gamma$, and χ is a parameter which essentially normalizes the spinor field. Now notice that if one sets $h^i = 0$ in the transformation laws (8) for G and G^*, then the restriction of these transformations to the subsets $\{X_a\}$ and $\{X_a, X_\alpha\}$ of the respective Lie algebras gives precisely the transformation laws for the adjoint representations $\{h^a\}$ and $\{h^a, h^\alpha\}$ of the groups $G_0 = \mathrm{SL}(2C)$ and $G_0^* = \mathrm{OSp}(1, 2C)$.

Moreover, the choice of Q_{AB} given above, with χ defined by

$$\epsilon_{ab} f_{\alpha\beta}{}^b = \chi f_{a\alpha}{}^\gamma (\gamma_5)_{\gamma\beta}, \tag{16}$$

corresponds to Clebsh-Gordan coefficients for an invariant product of two adjoint representations of G_0 and G_0^*, respectively. Hence, if in the actions I and I^* one sets $h_\mu{}^i = 0$, they become topological invariants I^0 and I^{*0}, the Euler-Poincaré characteristics of the manifolds with structure groups G_0 and G_0^*. This ensures the invariance of I^* under a supersymmetry transformation. In fact, the Euler-Lagrange equation that minimizes I^* under arbitrary variations of $h_\mu{}^a$ is $R_{\mu\nu}{}^{*i} = 0$, as can be inferred from an inspection of (15). It gives a constraint among the gauge potentials which can be solved for $h_\mu{}^a$. Insertion of this solution into the action leads to the second-order formalism for these theories. But then $\delta_\epsilon I^*$ given by (14) with $R_{\mu\nu}{}^{*i} = 0$ vanishes if $\epsilon^i = 0$. (Notice that the manner in which $h_\mu{}^a$ transforms under this variation is now irrelevant since the coefficient of $\delta_\epsilon h_\mu{}^a$ vanishes.) Therefore, an invariant supersymmetry transformation $\delta_\epsilon \alpha$ obtains in the second-order formalism.

Since the first- and second-order formalisms are completely equivalent, one can also derive an invariant supersymmetry transformation in the former. The transformation of the independent fields $h_\mu{}^a$ would be

$$\delta h_\mu{}^a = -\epsilon^\alpha f_{\alpha\beta}{}^a h_\mu{}^\beta - \tfrac{1}{2}\chi^{-1}\epsilon^{aij} h_i{}^\nu h_j{}^\lambda \epsilon^\alpha f_{k\alpha}{}^\beta (\gamma_5)_{\beta\gamma}(2h_\lambda{}^k R_{\mu\nu}{}^{*\gamma} + h_\mu{}^k R_{\lambda\nu}{}^{*\gamma})$$

with $f_{ij}{}^a = \frac{1}{4}\chi^2 \delta_{[ij]}{}^a$ and the inverse matrix $h_i{}^\nu$ being defined by $h_i{}^\nu h_\nu{}^j = \delta_i{}^j$. This is equivalent to the result of Townsend,[8] and analogous to that of Ref. 2, for the contracted version of this theory.

Denoting by $R_{\mu\nu}{}^0$ and $R_{\mu\nu}{}^{*0}$ the components of the curvature tensor for the spaces $P(G_0, M_4)$ and $P(G_0^*, M_4)$, we can write

$$R_{\mu\nu}{}^a = R_{\mu\nu}{}^{0a} + h_\nu{}^i h_\mu{}^j f_{ij}{}^a, \tag{17}$$

$$R_{\mu\nu}{}^{*a} = R_{\mu\nu}{}^{*0a} + h_\nu{}^i h_\mu{}^j f_{ij}{}^a = R_{\mu\nu}{}^a + h_\nu{}^\alpha h_\mu{}^\beta f_{\alpha\beta}{}^a, \tag{18}$$

$$R_{\mu\nu}{}^{*\alpha} = R_{\mu\nu}{}^{*0\alpha} + (h_\nu{}^i h_\mu{}^\beta - h_\mu{}^i h_\nu{}^\beta) f_{i\beta}{}^\alpha. \tag{19}$$

Then the actions I and I^* can be written as

$$I = I^0 + \int\{2R_{\mu\nu}{}^{0a} h_\lambda{}^j h_\sigma{}^i f_{ij}{}^b \epsilon_{ab} + h_\mu{}^i h_\nu{}^j h_\lambda{}^k h_\sigma{}^l f_{ij}{}^a f_{kl}{}^b \epsilon_{ab}\}\epsilon^{\mu\nu\lambda\sigma} d^4x, \tag{20}$$

$$I^* = I^{*0} + \int\{[2R_{\mu\nu}{}^{0a} h_\lambda{}^j h_\sigma{}^i f_{ij}{}^b \epsilon_{ab} + 4\chi R_{\mu\nu}{}^{*0\alpha} h_\lambda{}^\gamma h_\sigma{}^i f_{i\gamma}{}^\beta (\gamma_5)_{\alpha\beta}] - 4h_\nu{}^i h_\sigma{}^j h_\mu{}^\alpha h_\lambda{}^\beta f_{ij}{}^b f_{\alpha\beta}{}^a \epsilon_{ab} + h_\mu{}^i h_\nu{}^j h_\lambda{}^k h_\sigma{}^l f_{ij}{}^a f_{kl}{}^b \epsilon_{ab}\}\epsilon^{\mu\nu\lambda\sigma} d^4x, \tag{21}$$

where in deriving (21) we made use of (16).

The first terms I^0 and I^{*0} in (20) and (21) are topological invariants, the Euler-Poincaré characteristic of the manifold. The second term in (20) is an alternate form of Einstein's Lagrangian in the first-order formalism; the last term is the cosmological term. The brackets [ ] in (21) are an alternate form for the supergravity Lagrangian in the first-order formalism. The remaining terms are cosmological terms. If one subtracts I^0 and I^{*0} from (20) and (21), respectively, and makes a Wigner-Inonü contraction of the groups G and G^*, the cosmological terms will drop out. Then one recovers Einstein's theory and the supergravity action, respectively.

The variation of these actions with respect to arbitrary variations of the h's can be read off from the general expression (15). The Euler-Lagrange equations for the variational principle $\delta I = 0$ and $\delta I^* = 0$, are, respectively,

$$R_{\mu\nu}{}^i = 0; \quad R_{\mu\nu}{}^{[ij]} h_i{}^\nu h_j{}^\lambda = R_\mu{}^\lambda = 0, \tag{22}$$

and

$$R_{\lambda\sigma}{}^{*j} = 0, \tag{23}$$

$$\epsilon^{\mu\nu\lambda\sigma} h_\nu{}^i f_{i\beta}{}^\alpha R_{\lambda\sigma}{}^{*\beta} = 0, \tag{24}$$

$$\epsilon^{\mu\nu\lambda\sigma}[2h_\nu{}^j f_{ij}{}^b \epsilon_{ab} R_{\lambda\sigma}{}^{*a} + \chi h_\nu{}^\alpha f_{i\alpha}{}^\gamma (\gamma_5)_{\gamma\beta} R_{\lambda\sigma}{}^{*\beta}] = 0. \tag{25}$$

By introducing the notation $h_\nu{}^i f_{i\beta}{}^\alpha = (\gamma_\nu)_\beta{}^\alpha$ and $\eta^{ij} h_i{}^\nu f_{j\beta}{}^\alpha = (\gamma^\nu)_\beta{}^\alpha$, the last two equations can be rewritten as

$$(\gamma_\nu)_\beta{}^\alpha \tilde{R}^{*\beta\mu\nu} = 0 \ [\text{or } (\gamma^\nu)_\beta{}^\alpha R_{\mu\nu}{}^{*\beta} = 0], \qquad (26)$$

$$R_\mu{}^{*\lambda} = \tfrac{1}{4}\chi h_\nu{}^\alpha (\gamma_5 \gamma_\mu)_{\alpha\beta} \tilde{R}^{*\beta\lambda\nu}, \qquad (27)$$

where $\tilde{R}^{*\beta\mu\nu} = (1/2h)\epsilon^{\mu\nu\lambda\sigma} R_{\lambda\sigma}{}^{*\beta}$ and $h = \det(h_\mu{}^i)$.

The complete algebra of infinitesimal transformations that leave the action I^* invariant will be presented in a subsequent publication. This approach to gravitation and supergravity clearly exhibits the intimate connection between the two theories. Indeed, it becomes apparent that they have exactly the same geometric structure, differing only on the choice of gauge groups for the bundle spaces.

We remark that our formulation of the theories of gravitation and supergravity in the absence of matter fields is done completely in terms of the components of the curvature tensor without introducing a metric tensor, that is, it depends on the geometric properties of the affine bundle spaces. Of course, one can introduce a metric tensor $g_{\mu\nu} = h_\mu{}^i h_\nu{}^j \eta_{ij}$ which will be invariant under parallel transport. Such a tensor might actually be needed when matter fields are incorporated into the theory. It will be interesting to speculate on the possibility of extending this geometrical formulation of supergravity to spaces with more complex structures, including internal degrees of freedom for the matter fields.

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Threshold Electroproduction of Charged Pions from Light Nuclei

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Double-differential cross sections for the electroproduction of pions of both charges have been measured. We compare the data obtained for production near threshold from ^{12}C and ^{16}O with theoretical treatments employing both shell-model and sum-rule nuclear descriptions with full inclusion of the final-state interaction.

Recent results for low-energy elastic scattering of pions off nuclei[1] have confirmed that standard Kisslinger optical potentials,[2] used successfully to correlate the strong-interaction shifts and level widths of pionic atoms,[3] yield cross section predictions that disagree sharply with experiment. Although there is some indication that improved treatments of kinematic effects and short-range correlations may substantially reduce the discrepancies,[4] the need for further, independent probes of the pion wave function in nuclear matter is clear.

The modification of threshold electroproduction cross sections for complex nuclei by the final-state interaction of the emitted pion provides one such test of the Kisslinger potential. Experimen-