Thermodynamic Model for Composite-Particle Emission in Relativistic Heavy-Ion Collisions*

Aram Mekjian
Lawrence Berkeley Laboratory, Berkeley, California 94720, and Department of Physics, Rutgers University, New Brunswick, New Jersey 08904
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A thermodynamic model, constructed in the framework of the big-bang equilibrium model, is proposed and used to study the spectra of high-energy composite particles that can be seen in relativistic heavy-ion collisions. Properties of the spectra are related to properties of the thermodynamic equilibrium region.

In this Letter, properties of the spectra of high-energy composite particles emitted in a relativistic heavy-ion collisions[1,2] are studied in the framework of a thermodynamic picture whose foundation is based on the big-bang equilibrium model.[3] The basis of the model to be discussed is the nuclear fireball picture used to describe proton inclusive spectra.[2] In this picture, nucleons mutually swept out from the combined system of target and projectile form an equilibrated fireball which then expands freely. The results of this model[2] fit the gross features of the proton inclusive spectra for 400-MeV/nucleon Ne^{20} and He^4 on uranium for proton energies above 80 MeV.

On the other hand, the composite-particle spectra seen in the same experiments have been interpreted in terms of a model in which nucleons with small relative momenta coalesce.[1,4] Specifically, this model imposes a momentum-space restriction for formation of composite particles; and this restriction, in turn, leads to correlations in energy and angle between double differential cross sections of composite particles and powers of the corresponding proton cross sections. These features are borne out in the experimental results as shown in the figures of Ref. 1. Now, within the framework of the coalescence model,[1] no explicit reference has been made to the spatial evolution of the cascade nucleons and to the possible equilibration properties that they may have. It is this aspect and its consequences that will be investigated in this paper.

Since a detailed description of an expanding collection of strongly interacting nucleons raised to a high temperature (the fireball) is impossible, simplifying assumptions or idealizations have to be made. Here, the idealization is based on the three phases that this collection goes through in its expansion. First, when densities and temperatures are high, mean free paths are short ($l_0 \approx 1$ fm or less), and collisions are then frequent in the bulk of the material, causing scattering to all possible states (continuum and bound), with composite-particle formation and break-up having its equal place in the system. In this phase, a temporal thermodynamic equilibrium could perhaps be established. In a second stage as the collection of these nucleons and nuclei expands, nonequilibrium few-collision processes dominate until the density is so low that a third phase is reached in which all collisions cease and the gas expands freely.

Now, as a working idealization, we replace this complicated evolution with a much simpler one in which the fireball expands through a set of equilibrium states until a volume V_0 or density $\tilde{\rho}$ is reached, after which all collisions cease instantaneously. The system then expands freely. This idealization leads to a framework for the model that is analogous to that of the big-bang

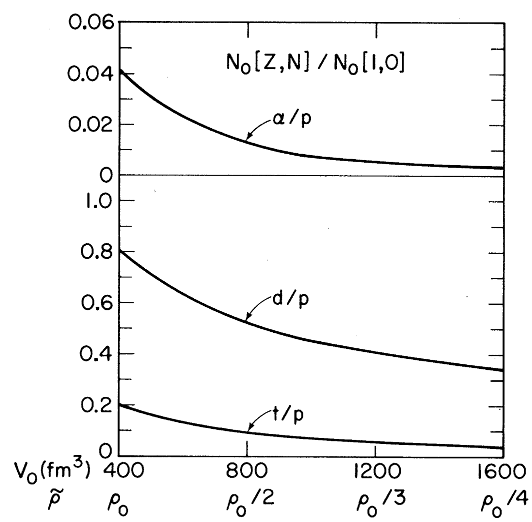

FIG. 1. The ratio of d,t,α to p as a function of the thermodynamic volume V_0 or density $\tilde{\rho}$. The evaluation is for $kT_0 = 50$ MeV and for $\tilde{Z} = 30$, $\tilde{N} = 30$, which are typical values in Ref. 2. For sixty nucleons, normal nuclear matter with density $\rho_0 = 0.15$ nucleons/fm^3 occupies a volume 400 fm^3. From the figure, we note the persistence of a large d/p ratio even in diffuse regions. A small observed d/p ratio could indicate a diffuse $\tilde{\rho}$. A large α/p ratio could indicate a dense $\tilde{\rho}$.

equilibrium model.[3] Here, no attempt is made to justify the model, but only to explore its consequences and its possible relevance to relativistic heavy-ion collisions.

Under the assumption that thermodynamic equilibrium is established in a volume V_0 and at a temperature T_0, the number of nuclei $N_0[Z,N]$ in equilibrium is determined by statistical factors alone and is[3]

$$N_0[Z,N]=\left(\frac{[\lambda(T_0)]^3}{V_0}\right)^{A-1}\frac{f(Z,N)}{2^A}(N_0[1,0])^Z(N_0[0,1])^N, \tag{1}$$

where $\lambda(T)=hc(2\pi m_p c^2kT)^{-1/2}$, $A=Z+N$, and

$$f(Z,N)=A^{3/2}\exp[E_0(Z,N)/kT_0]\sum_j(2S_j+1)\exp(-E_j/kT_0). \tag{2}$$

The above equations are applicable in the high-temperature and low-density limit. The summation is over the ground and excited states of the nucleus (Z,N), with S_j being the spins of these states and E_j being their excitation energy measured from the ground-state energy $E_0(Z,N)$. $N_0[1,0]$ and $N_0[0,1]$ are, respectively, the number of protons and neutrons in V_0 at equilibrium. If we define $\tilde{Z}$ and $\tilde{N}$ as the initial numbers of protons and neutrons, respectively, in the fireball, the following auxiliary conditions must be satisfied:

$$\sum_{Z,N} N_0[Z,N]=\tilde{Z};\quad \sum_{Z,N} NN_0[Z,N]=\tilde{N}. \tag{3}$$

It is worthwhile to point out that the high temperature involved ($kT_0=50$ MeV from inclusive proton spectra[2]) greatly weakens the dependence of the abundance on the traditional Boltzmann factor. An example of the abundance ratios is shown in Fig. 1 and discussed in the figure caption.

Next, the momentum distribution of the individual species of this composite gas is given by the Maxwell-Boltzmann distribution in the rest frame of the fireball; thus,

$$d^3N_0[Z,N;\vec{P}]/d^3P=N_0[Z,N]\exp(-E_K/kT_0)(2\pi Am_p kT_0)^{-3/2}, \tag{4}$$

where $E_K=P^2/2m_A$ is the total kinetic energy of the nucleus $[Z,N]$ whose abundance $N_0[Z,N]$ is determined by Eq. (1).

At this point a formal correspondence with the results of the coalescence model can be made. In this model, the momentum phase-space restriction that composite particles be constructed from nucleons whose momenta lie within a sphere of radius P_0 of each other leads to[1]

$$\frac{d^2n[Z,N]}{P_n{}^2dP_n\,d\Omega}=\frac{2S_A+1}{2^A}\frac{1}{Z!N!}R_{\mathrm{p,t}}{}^N\left(\frac{4\pi}{3}\gamma P_0{}^3\right)^{A-1}\left(\frac{d^2n[1,0]}{P_n{}^2dP_n\,d\Omega}\right). \tag{5}$$

The P_n is the momentum per nucleon, $d^2n[Z,N]/P_n{}^2dP_n\,d\Omega$ is the number of nuclei $[Z,N]$, or protons $[1,0]$, per event per unit element of phase space; $\gamma=(1+P_n{}^2/m_p{}^2)^{1/2}$ and $R_{\mathrm{p,t}}=(N_\mathrm{p}+N_\mathrm{t})/(Z_\mathrm{p}+Z_\mathrm{t})$, where the target is $[Z_\mathrm{t},N_\mathrm{t}]$ and projectile is $[Z_\mathrm{p},N_\mathrm{p}]$. In Eq. (5) we have explicitly included the spin-alignment factor $(2S_A+1)/2^A$ between the individual nucleons and the composite $[Z,N]$ (implicit in Ref. 1). Now, it is interesting to note that the result of Eq. (5), relating the momentum-space density of the composite system to powers of the proton density, is also a feature of the equilibrium model. Specifically, Eq. (4) can be rewritten as

$$\frac{d^2N_0[Z,N;\vec{P}]}{A^3P_n{}^2dP_n\,d\Omega}=\frac{f(Z,N)}{A^{3/2}2^A}R_0{}^N\left(\frac{h^3}{V_0}\right)^{A-1}\left(\frac{d^2N_0[1,0;P_n]}{P_n{}^2dP_n\,d\Omega}\right)^A, \tag{6}$$

where $P_n{}^2/2m_p=E_K/A$ and where $R_0=N_0[0,1]/N_0[1,0]$. In obtaining Eq. (6), we have used the result $A^3P_n{}^2dP_n=P^2dP$, where $\vec{P}$ is the total momentum. Thus, the equilibrium model has all the essential features of the coalescence model since Eq. (6) is formally similar to Eq. (5). However, a difference exists (apart from combinatorial and phase-space factors), which is that the proton density in momentum space in the thermodynamic model is fixed by Eq. (4) to be the equilibrium thermal distribution.

Next, the double differential cross sections for protons and composite particles can be obtained as follows. For each impact parameter b (suppressed until now), the number of nucleons in the fireball generated by the geometrical sweeping of the projectile over the target, the velocity β of the fireball, and the energy available per nucleon ϵ in the center-of-mass system of the fireball can be calculated as outlined in Ref. 2. This procedure defines a set of quantities $\tilde{Z}(b)$, $\tilde{N}(b)$, $\beta(b)$, and $\epsilon(b)=\frac{3}{2}kT_0(b)$. Then from Eqs. (1)–(3), the values of $N_0[Z,N;b]$ can be evaluated as a function of $V_0(b)$. The $V_0(b)$ should satisfy $[\tilde{Z}(b)+\tilde{N}(b)]/V_0(b)=\tilde{\rho}$, the thermodynamic density, which is to be taken as independent of b. The momentum-space density in the rest system of the fireball then follows from Eq. (4). This result can subsequently be transformed to the laboratory system and from p^2dp space to dE space. The double differential cross sections follow when the resulting expression is integrated over impact parameters.

The above scheme outlines a well-defined operational procedure for calculating double differential cross section of composite particles in a specific model. An interesting feature of the model arises when the weak impact-parameter dependences of $T_0(b)$, $\beta(b)$, and $N_0[0,1;b]/N_0[1,0;b]\equiv R_0(b)$ are neglected. In this approximation, the composite-particle cross section can be written as

$$\frac{d^2\sigma[Z,N]}{dtd\Omega}=A^{3/2}\frac{f(Z,N)}{2^A}\bar{R}_0{}^N\left(\frac{h^3\tilde{\rho}_p}{G_0}\right)^{A-1}[f(t,\theta)]^{A-1}\left(\frac{d^2\sigma[1,0]}{dtd\Omega}\right)^A. \tag{7}$$

Here,

$$[f(t,\theta)]^{-1}\equiv[t(t+2m_p)]^{1/2}\bar{\gamma}\{t+m_p-\bar{\beta}[t(t+2m_p)]^{1/2}\cos\theta\}$$

with t the kinetic energy per particle, $G_p\equiv\int 2\pi bN_0[1,0;b]\,db$, and $\tilde{\rho}_p=N_0[1,0;b]/V_0(b)$; $\bar{T}_0$, $\bar{\beta}$, $\bar{\gamma}=(1-\bar{\beta}^2)^{-1/2}$, and $\bar{R}_0$ are the constant values of these quantities. The result of Eq. (7) is of the form of the coalescence result, [Eq. (2) of Ref. 1]. It thus contains the same correlation in energy and angle between composite-particle cross sections and powers of the proton cross sections. However, some departure in correlation is introduced when the impact-parameter dependence of T_0 and R_0 are considered. Moreover, the result allows a formal identification

$$\left(\frac{4\pi P_0{}^3}{3\sigma_0}\right)^{A-1}R_{t,p}{}^N\frac{1}{Z!N!}\leftrightarrow A^3R_0{}^N\exp(E_0/kT_0)\left(\frac{h^3\tilde{\rho}_p}{G_p}\right)^{A-1}, \tag{8}$$

where σ_0 is the total reaction cross section.

In Eq. (7), the proton cross section is not to be taken from experiment, but is to be calculated by use of the procedure just outlined. Thus, in energy regions where the calculated proton cross sections differ from the experimental results,[2] we can expect, in general, that the calculated composite-particle cross sections will also differ from experiment, since they are correlated to powers of the proton cross sections. Consequently, a detailed comparison of the predictions of this model with experiment will not be made at this time since the composite-particle cross-section data[1] are in energy regions (less than 100 MeV/nucleon) where large departures from a clean high-temperature thermal spectrum (fireball spectrum) for protons are present. These departures are believed to be due to the remnants of the target contributing to the spectra.[5] Moreover, the use of the identification given in Eq. (8) to obtain properties of the fireball density at "freeze-out" through the extracted P_0's of Ref. 1 can also be misleading since the P_0's of Ref. 1 refer to the composite-particle spectra for energies less than 100 MeV/nucleon. Nevertheless, it is interesting to pursue this correspondence further. With use of the extracted P_0's of Ref. 1 for the 400-MeV/nucleon Ne^{20} and U data, values of $\tilde{\rho}_z$ follow when use is made of the approximate scaling $\tilde{\rho}_Z/G_Z\approx\tilde{\rho}_p/G_p$ so that $\tilde{\rho}_Z=\tilde{Z}(b)/V_0(b)$ and $G_Z=\int 2\pi b\tilde{Z}(b)db=4800$ fm^2. The resulting values depend on the composite particle and are $\tilde{\rho}_z/\rho_1\cong\frac{2}{3}$, $\frac{1}{3}$, and $\frac{1}{5}$ for α, t, and d, respectively, where $\rho_1=0.075$ protons/fm^3 is the nuclear matter density of protons. If the above values of P_0 were appropriate in the energy range above 100 MeV/nucleon, then the conclusion could be drawn that there does not exist a single set of equilibrium conditions that would consistently explain all the cross sections. It is interesting to note that these "freeze-out" densities are slightly less than the central density of the corresponding composite system. Thus, if higher-energy composite-particle data result in the same values of P_0 given above, a model could perhaps be developed which incorporates the correlation between the "freeze-out" density and the central density of the composite.

In summary, a thermodynamic model whose foundation is based on the equilibrium big-bang model is investigated to see whether it has any relevance to the composite-particle spectra that can be seen in relativistic heavy-ion collisions.

The model, if valid, would best be applicable to the energy region in the spectrum of a composite particle above 100 MeV/nucleon. Unfortunately, no data exist at present in the appropriate range for the composite particles. It therefore would be interesting to have such data to see whether a high-energy thermal component is present, and if so, to use it to extract the thermal properties of the fireball.

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Proof that the H^- Ion Has Only One Bound State

Robert Nyden Hill
Physics Department, University of Delaware, Newark, Delaware 19711
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It is rigorously demonstrated that the nonrelativistic H^- ion has only one bound state in the fixed (infinite-mass) nucleus approximation with Coulomb interactions only.

The H^- ion, made up of a proton and two electrons, has long been known to have one bound state.[1] Additional bound states have never been found, but their nonexistence has so far not been proved. The present note provides the proof in the fixed (infinite-mass) nucleus approximation with Coulomb interactions only. The importance of the present result stems from the qualitative difference between the bound-state spectrum of negative ions (of which H^- is the simplest example) and the bound-state spectrum of positive ions and neutrals. Negative ions have only a finite number of bound states,[2] for which correlation effects are decisive (H^-, for example, is believed to have no bound states in Hartree-Fock approximation). Positive ions and neutrals, on the other hand, have an infinite number of bound states.[3]

The nonrelativistic Schrödinger equation for two electrons interacting with each other and with a fixed nucleus of charge Ze via Coulomb forces can be written in the form $H|\psi\rangle = E|\psi\rangle$, where $H = H_0 + V$, with

$$H_0(\vec{r}_1, \vec{r}_2; \vec{r}_1{}', \vec{r}_2{}') = (-\nabla_1{}^2 - 2Zr_1{}^{-1} - \nabla_2{}^2 - 2Zr_2{}^{-1})\delta(\vec{r}_1 - \vec{r}_1{}')\delta(\vec{r}_2 - \vec{r}_2{}') \tag{1}$$

and

$$V(\vec{r}_1, \vec{r}_2; \vec{r}_1{}', \vec{r}_2{}') = 2|\vec{r}_1 - \vec{r}_2|^{-1}\delta(\vec{r}_1 - \vec{r}_1{}')\delta(\vec{r}_2 - \vec{r}_2{}'). \tag{2}$$

Atomic units have been used and continuous matrix notation adopted for later convenience.

Proving the nonexistence of bound states requires a method which provides lower bounds to energy eigenvalues. The basic tool to be used here is a well-known comparison theorem.[4]

Theorem 1.—Let $H^{(1)}$ and $H^{(2)}$ be two Hermitian Hamiltonians whose discrete eigenvalues below the continuum can be characterized by the familiar variational principle $E = \min\langle\psi|H|\psi\rangle\langle\psi|\psi\rangle^{-1}$, with the minimization for excited states carried out subject to the constraint that $|\psi\rangle$ be orthogonal to preceding eigenvectors. Denote the ordered eigenvalues of $H^{(i)}$ by $E_1{}^{(i)} \leq E_2{}^{(i)} \leq \ldots \leq E_n{}^{(i)} \ldots \leq E_c{}^{(i)}$, where E_c is the energy at which the continuous spectrum (if any) begins. Then if $\langle\psi|H^{(1)}|\psi\rangle \leq \langle\psi|H^{(2)}|\psi\rangle$ holds for all admissible state vectors $|\psi\rangle$, $E_n{}^{(1)} \leq E_n{}^{(2)}$ holds for all n, and $E_c{}^{(1)} \leq E_c{}^{(2)}$. The result of the present paper will be obtained from theorem 1 by letting $H^{(2)} = H = H_0 + V$ while $H^{(1)}$ is something more tractable.

The lower-bounding Hamiltonian $H^{(1)}$ will be constructed by generalizing a method introduced by Bazley[5] to construct lower bounds to helium eigenvalues: Replace V in $H^{(2)} = H = H_0 + V$ by