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Phase Transition in an Ising Model near the Percolation Threshold*

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The phase transition near the percolation threshold in an Ising model with random exchange is investigated. The mean-field theory of the transition is discussed. The crossover exponent φ and susceptibility are caluclated in $6-\epsilon$ dimensions using renormalization-group methods with the result $\varphi = 1 + O(\epsilon^3)$. φ is also shown to be 1 near one dimension.

We consider an Ising model in which the nearest-neighbor exchange interaction has a probability p and $1-p$ of taking on the values J and 0, respectively. As p is decreased the critical temperature $T_c(p)$ decreases and reaches zero at the percolation point p_c. In this Letter we study the properties of the phase transition at low temperatures and near p_c. A sketch of the phase diagram is given in Fig. 1. The two physical variables entering this problem are conveniently taken to be $r_0 \sim p_c - p$ and $w \sim e^{-2\beta J}$. The scaling fields μ_2 and μ_1 are determined by the special directions in the phase diagram in the renormalization-group sense. These directions are tangent to the critical line at p_c and along the $T=0°$ axis, respectively.[1] Along the $T=0°$ axis, the magnetic properties are determined by the properties of the connected spin clusters; i.e., it is a percolation problem.[2] Along the critical line away from p_c critical behavior appropriate for a random system occurs.[3] Near p_c and at low temperatures there is a competition between these two effects. Stauffer[4] and others[5,6] have argued that the point $p=p_c$, $T=0°$ should be viewed as a multicritical point. This is discussed further below. A magnet with random occupation of the sites, $Rb_2Mn_pMg_{1-p}F_4$, has recently been studied experimentally near the percolation point by Birgeneau *et al.*[7]

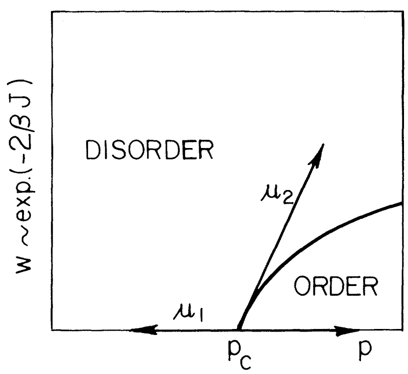

FIG. 1. Phase diagram of a random Ising model. The physical fields are $r_0 \sim p_c - p$ and $w \sim e^{-2\beta J}$. The scaling fields $\mu_2 \sim w$ and $\mu_1 \sim r_0 + aw$ are determined by the special directions.

In order to facilitate averaging over the quenched random exchange interaction we consider n identical replicas of the Ising model.[8] After averaging over the exchange interaction, the partition function for the n replicas is

$$Z(n) = \mathrm{Tr} \prod_{\mathrm{nn}} \{1 + v \exp[\beta J \sum_{\alpha=1}^{n} (\mu_\alpha{}^i \mu_\alpha{}^j - 1)]\} \exp(h \sum_{i} \sum_{\alpha} \mu_\alpha{}^i), \tag{1}$$

where $v = P/(1-p)$, $h = \beta H$, where H is the magnetic field, the Ising spin $\mu_\alpha{}^i$ on site i for replica α takes on the values ± 1, and the product is over all nearest-neighbor pairs. The required partition function is then given by

$$\langle \ln Z\rangle = \lim_{n\to 0} n^{-1}\ln Z(n).$$

We rewrite (1) in the form $Z(n) = \mathrm{Tr}_{\mu} e^{-\beta H}$. This is done by introducing projection operators

$$P_0^{ij} = \prod_{\alpha} \tfrac{1}{2}(1+\mu_\alpha{}^i\mu_\alpha{}^j), \quad P_1^{ij} = \sum_{\alpha} \tfrac{1}{2}(1-\mu_\alpha{}^i\mu_\alpha{}^j)\prod_{\beta}{}' \tfrac{1}{2}(1+\mu_\beta{}^i\mu_\beta{}^j), \text{ etc.} \tag{2}$$

The Hamiltonian is then

$$-\beta H = \sum_{\mathrm{nn}} [A_0 P_0^{ij} + A_1 P_1^{ij} + \ldots A_n P_n^{ij}] + h \sum_{i}\sum_{\alpha} \mu_i{}^\alpha, \tag{3}$$

where

$$A_0 = \ln[1/(1-p)], \quad A_s = \ln(1 + ve^{-2s\beta J}) \simeq ve^{-2s\beta J}. \tag{4}$$

The correct low-temperature limit is obtained by letting $n\to 0$ after letting $T\to 0$, and thus at low temperatures we only need retain A_0 and A_1. We are thus led to consider an Ising model in which at each site we have n spins μ_α. It is convenient to regard this as a model in which each site can be in one of 2^n states.

To discuss the disordered phase we introduce a complete set of functions appropriate for a disordered system: μ_α, $\mu_\alpha\mu_\beta$ $(\alpha<\beta)$, $\mu_\alpha\mu_\beta\mu_\gamma$ $(\alpha<\beta<\gamma)$, etc. We use a notation $\mu_{(\alpha)}$ for all these functions. The index (α) takes on all the values α, $\alpha\beta$, $\alpha\beta\gamma$, etc., arranged in increasing order. The Hamiltonian is then given by[9] (omitting a constant)

$$-\beta H = \sum_{\mathrm{nn}} [K_1 \sum_{\alpha} \mu_\alpha{}^i\mu_\alpha{}^j + K_2 \sum_{\alpha<\beta} \mu_\alpha{}^i\mu_\beta{}^i\mu_\alpha{}^j\mu_\beta{}^j \ldots] + h \sum_{i}\sum_{\alpha} \mu_\alpha{}^i, \tag{5}$$

where $2^n K_s = \ln[1/(1-p)] + (n-2s)ve^{-2\beta J}$. In the mean-field theory of the transition, we introduce order parameters $m_1 = \langle\mu_\alpha\rangle$, $m_2 = \langle\mu_\alpha\mu_\beta\rangle$, etc.[10] In the disordered phase $m_1 \sim h$, $m_2\sim h^2$ etc., so that to order h^2 we only need consider m_1 and m_2. Similarly along the critical line $m_1 \sim (T_c - T)^{1/2}$, $m_2 \sim (T_c - T)$ etc., so again we only need m_1 and m_2. However, in the ordered phase near the $T = 0°$ axis, all the order parameters are of the same magnitude and this approximation only gives qualitative results. Near p_c, the mean-field equations are (for $n=0$)

$$m_1(r_0 + w) = h - m_1 m_2, \quad m_2(r_0 + 2w) = m_1{}^2 - 2m_2{}^2, \tag{6}$$

where $2^n r_0 = z(p_c - p)/(1-p_c)$, $2^n w = 2zve^{-2\beta J}$, and z is the number of nearest neighbors. In the disordered phase $m_1 = h/(r_0+w)$, $m_2 = h^2(r_0+w)^{-2}(r_0+2w)^{-1}$. The spontaneous magnetization[11] near the critical line is given by $m_{10} = |r_0+w|^{1/2}(|r_0+w|+w)^{1/2}$. The mean-field critical line is $r_0 + w = 0$ and this expression shows the crossover behavior: Choose variables $\mu_1 = r_0 + w$, w, and $m_{10} = |\mu_1|(1 + w/\mu_1{}^{1/\varphi})^{1/2}$ with $\varphi = 1$. Near the critical line $\omega \gg |\mu_1|$, $m_{10} \sim |\mu_1|^{1/2}$ while along the $\omega = 0$ axis $m_{10} = |\mu_1| = |r_0|$. This latter result is only qualitatively correct (the correct result is $2|r_0|$ see below). Results equivalent to these have been given by Bidaux, Carton, and Sarma[9] using a variational method.

To discuss the ordered phase we introduce functions appropriate for an ordered state:

$$P_0{}^i = \prod_{\alpha} \tfrac{1}{2}(1+\mu_\alpha{}^i), \quad P_\alpha{}^i = \tfrac{1}{2}(1-\mu_\alpha{}^i)\prod_{\beta}{}' \tfrac{1}{2}(1+\mu_\beta{}^i), \text{ etc.} \tag{7}$$

We also use the notation $p_{(\alpha)}$ for these functions where (α) can also take on the value 0. The Hamiltonian is $(n=0)$

$$-\beta H = \sum_{\mathrm{nn}} [A_0 \sum_{(\alpha)} p_{(\alpha)}{}^i p_{(\alpha)}{}^j + A_1 \sum_{\alpha}\sum_{(\beta)} (p_{\alpha(\beta)}{}^i p_{(\beta)}{}^j + p_{(\beta)}{}^i p_{\alpha(\beta)}{}^j)] - 2h\sum_{i}[\sum_{\alpha} p_\alpha{}^i + 2\sum_{\alpha<\beta} p_{\alpha\beta}{}^i + \ldots]. \tag{8}$$

In the mean-field theory, we introduce order parameters $x_0 = 1 - \langle p_0\rangle$, $x_1 = 1 - \langle p_\alpha\rangle$, etc. The equations satisfied by these order parameters near p_c are

$$x_0 = 0, \quad r_0 x_1 + \tfrac{1}{2}x_1{}^2 + \tfrac{1}{2}wx_2 = 2h, \quad r_0 x_s + \tfrac{1}{2}x_s{}^2 + \tfrac{1}{2}ws(x_{s+1} - x_{s-1}) = 2sh. \tag{9}$$

Along the $w=0$ axis these equations separate and the spontaneous order $x_{s0} = 2|r_0|$. The magnetization is also $m_{10} = 2|r_0|$. If $w < |r_0|$ we only need consider the first few of Eq. (9): $x_{10} = 2|r_0| - w$, $x_{20} = 2|r_0|$

$-w^2/|r_0|$.

Critical exponents in $6-\epsilon$ dimensions can be calculated by writing the partition function as a Gaussian integral. We introduce a variable $s_{(\alpha)}$ conjugate to each $\mu_{(\alpha)}$ in (5) and the effective Hamiltonian is

$$\beta H_E = \tfrac{1}{2}\sum_{(\alpha)}\sum_k |s_{(\alpha)k}|^2(r_{0(\alpha)}+k^2) - hN^{1/2}\sum_\alpha s_{\alpha k=0} - \tfrac{1}{6}uN^{-1/2}\sum_{(\alpha)(\beta)(\gamma)}{}' \sum_k s_{(\alpha)k_1}s_{(\beta)k_2}s_{(\gamma)-k_1-k_2}, \qquad (10)$$

where N is the number of lattice sites and $r_{0\alpha}=r_{01}=r_0-(\frac{1}{2}n-1)w$, $r_{0\alpha\beta}=r_{02}=r_0-(\frac{1}{2}n-2)w$, etc. The sum in the third term is over all ordered indices $(\alpha),(\beta),(\gamma)$ such that each replica 1, 2, ..., n either appears twice (i.e., once in each of two different indices) or not at all. The recursion relations[12] are obtained by integrating over all fluctuations with wave vectors $b^{-1}<k<1$, rescaling all lengths by a factor of b^{-1} and scaling $s\to\zeta s$, where ζ is chosen so that the k^2 term remains unchanged:

$$u' = ub^{(\epsilon-3\eta)/2}[1+2\bar{u}^2(2^n-3)\ln b], \qquad (11)$$

$$r_{0s}' = b^{2-\eta}\left\{r_{0s}[1-2\bar{u}^2\ln b] + 2\bar{u}^2\ln b\sum_{t=1}^{n}\binom{n}{t}r_{0t}\right\}, \qquad (12)$$

where $\eta=\frac{1}{3}\bar{u}^2(2^n-2)$ and $\bar{u}^2=u^2V/(2^7\pi^3N)$. Terms which vanish when $b\to\infty$ have been omitted in (12). Equation (11) is exactly the same as that for the 2^n-component Potts[13] model and the fixed point value $\bar{u}^{*2}=\epsilon/(10-3\times2^n)$. We discuss Eq. (12) in some special cases. (a) For $n=1$, our model is just the Ising model and Eq. (12) reduces to one equation $r_{01}'=b^2r_{01}$, which leads to classical exponents as expected near $d=6$. (b) For general $n>1$, we have $n-1$ degenerate eigenvectors $r_{0s+1}-r_{0s}=w$ with recursion relation

$$w'=b^y w, \quad y=2-(\epsilon/3)(2^n+4)(10-3\times2^n)^{-1}, \qquad (13)$$

and one nondegenerate eigenvector

$$R=\sum_{t=1}^{n}\binom{n}{t}r_{0t}=(2^n-1)r_0+\tfrac{1}{2}nw,$$

with recursion relation

$$R'=b^xR, \quad x=1/\nu=2+(5\epsilon/3)(2^n-2)(10-3\times2^n)^{-1}, \qquad (14)$$

where ν is the exponent for the correlation length. The crossover exponent

$$\varphi=x/y=1+(2^n-1)\epsilon/(10-3\times2^n)^{-1}, \quad n\neq1. \qquad (15)$$

(c) For $n=0$, $\sum_t\binom{n}{t}r_{0t}\equiv0$ and $r_{0s}'=b^{(1/\nu_p)}r_{0s}$, where ν_p is the correlation-length exponent for the percolation problem[13] ($\nu_p=\frac{1}{2}+5\epsilon/84$). There are n degenerate eigenvectors r_{0s}. We have shown that this degeneracy remains to $O(\epsilon^2)$ so that the crossover exponent for the random magnet is $\varphi=1+O(\epsilon^3)$.

The susceptibility in the disordered phase near p_c can be written in the form suggested by Stauffer,[4] $\chi^{-1}=A\mu_1^{\gamma_p}f(w/\mu_1^{1/\varphi})$ where A is chosen so that $f(0)=1$, $\gamma_p=1+\frac{1}{7}\epsilon$ is the percolation exponent. The scaling fields $\mu_1=r_0+aw$, $\mu_2=w$ are chosen as in Fig. 1 so that a determines the slope of the critical line at p_c. The function f vanishes on the critical line with an exponent γ. For $n=0$ and to order ϵ, f is given by

$$f(x)=1-\tfrac{1}{2}\bar{u}^{*2}+x(1-a)-\bar{u}^{*2}x^{-1}\sum_{s=1}^{\infty}(-1)^s[g_{s+1}(x)-g_s(x)], \qquad (16)$$

where $g_s(x)=[1+x(s-a)]^2\ln[1+x(s-a)]$. As $\varphi=1$, the critical line is $\mu_1=0$ and we require that f vanish on this line. This condition determines $a=1-\frac{7}{2}\pi^2\bar{u}^{*2}\zeta(3)$, where ζ is the Riemann zeta function. Then for large x, $f(x)\sim x^{(\gamma_p-1)}$ and we obtain mean-field behavior near the critical line as expected near six dimensions.

It is of interest to compare our results with a heuristic picture (due to de Gennes[14]) discussed by Lubensky[5] and Stanley *et al.*[5] In this picture the connected spin clusters near p_c are regarded as a collection of nodes connected by links which are approximated by self-avoiding random walks. This picture predicts that $\nu_p\leq\varphi^{-1}\leq\nu_p/\nu_s$ where ν_p is given above and ν_s is the correlation-length exponent for the self-avoiding random walk. For $d\geq6$, $\varphi^{-1}=\nu_p/\nu_s=1$, but for $d=6-\epsilon$, $\nu_p=\frac{1}{2}+5\epsilon/84$, $\nu_s=\frac{1}{2}$, and $\varphi^{-1}<\nu_p/\nu_s$. It is unreasonable to extrapolate our result to low dimensions. For this reason, we calculated φ for our model using Migdal's[15] method, which should be accurate near one dimension, with the result $\varphi = 1 + O[(d-1)^2]$. This agrees with a result of Kirkpatrick[16] obtained by Migdal's method without using the replication device. The experimental results[7] on a Heisenberg magnet with sit randomness in two dimensions give $\varphi^{-1} \sim 1.7$. This difference may reflect the difference between Heisenberg and Ising systems.

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COMMENTS

Testing Time-Reversal Invariance in Inclusive Single-Lepton and Dilepton Neutrinoproduction

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The leptonic correlations pointed out by Sachs can serve as tests of time-reversal invariance under reasonable assumptions in inclusive single-lepton production by neutrinos. For dilepton production, the presence of strong hadronic final-state interactions does not allow the corresponding conclusion.

This Comment is devoted to an examination of the conditions under which the purely leptonic correlations recently pointed out[1] by Sachs can serve as tests of time-reversal invariance (henceforth called T invariance). I consider inclusive lepton production by neutrinos ν (taken to be of the muon type); the corresponding arguments of course apply for antineutrino projectiles. Under "General considerations" I define the question: the derivation, under T invariance, of the equality (1.3) between cross sections for configurations which are related by a time reversal of the detected leptonic variables; because of Eq. (1.3), the correlations of Sachs would then indicate T noninvariance. Then I derive (1.3) for single-lepton production, and compare it with the derivation of Ref. (1). I then discuss dilepton production, and in particular, the difficulty in deriving (1.3) for this case in the Sachs model.

A particle symbol will denote also its various characteristics (momentum, spin, charge, etc.); the tilde over a particle symbol means the time