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Theory of Turbulence in Superfluid ^{4}He

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A consideration of the dynamics of a vortex tangle leads to a new equation describing turbulence in superfluid helium. The equation is seen to be remarkably successful in predicting the steady-state properties of dissipative counterflow.

At sufficiently small velocities, superfluid helium will flow through a channel without any measurable dissipation. Above certain critical velocities, however, dissipative behavior sets in as small amounts of quantized vortex line grow by interacting nonconservatively with the normal fluid and with the walls of the channel. Although interest has centered primarily on the critical velocities themselves, numerous experiments have also been performed to study the turbulent state generated when the superfluid is driven far into the dissipative regime.[1-5]

It was suggested by Vinen in his admirable papers on the subject that steady-state superfluid turbulence will consist of a random tangle of quantized vortex lines maintained in equilibrium by competing growth and annihilation processes. He proposed an equation governing the total line length L per unit volume in the presence of counterflowing normal and superfluid velocity fields $\vec{U}_n$ and $\vec{U}_s$. Although this equation has proved extremely useful, it is essentially phenomenological in character. That is, the theoretical arguments which were used to derive the Vinen equation were based on a number of erroneous premises, the most important of which was that the important characteristic velocity acting on the vortex tangle is the random interline velocity. Our purpose here is to present a new theory of superfluid turbulence, obtained by considering the actual dynamics of a vortex tangle. The predictions of the theory are compared with the determination by Vinen[1] of L as a function of $U_n - U_s$ and T, and the recent measurements by Ashton and Northby[5] of the average drift velocity of the vortex tangle. Additional calculations, as well as details of the derivation and of the numerical integration technique, will be given in later papers.

The vortex tangle must be treated in some approximate statistical fashion. If we describe the vortex line by the parametric form $\vec{r}(s,t)$, the local self-induced velocity $\partial\vec{r}(s,t)/\partial t$ measured with respect to U_s is given by[6]

$$\vec{v}_1 = -(\kappa/4\pi)\vec{r}' \times \vec{r}''[\ln(ar'') + O(1)], \qquad (1)$$

where primes denote differentiation with respect to the arc length s, $r'' = |\vec{r}''|$, κ is the quantum of circulation, a is the core cutoff parameter, and $O(1)$ represents nonlocal corrections of order 1. Since the behavior of a local line element is determined primarily by this velocity, we characterize the vortex tangle in terms of a distribution in $\vec{v}_1$, or, equivalently, a distribution in $\vec{v}_1/v_1$ and the local radius of curvature $R = (r'')^{-1}$. Although it seems to be a reasonable simplification to consider the distribution to be spatially homogeneous,[7] it will certainly depend on the angle θ between $\vec{v}_1$ and the axis of the counterflow (taken to be positive in the direction defined by $\vec{U}_n$). As the simplest physically interesting case we therefore consider the distribution function $\lambda(R,\theta,t)$, where $\lambda(R,\theta,t)2\pi R^2 \sin\theta\, dR\, d\theta$ is the line length per unit volume with local radius of curvature in the range R to $R+dR$, and $\vec{v}_1$ heading into the range θ to $\theta+d\theta$.

The vortex tangle will develop in time as each line element moves under the combined effects of normal fluid scattering, local self-induced motion, and the velocity fields arising from other line elements in its neighborhood. Vinen[1] and other authors have already shown that the nonconservative action of the normal fluid on a line element dl gives rise to the rates of change

$$\dot{R} = \alpha[(U_n - U_s)\cos\theta - \beta/R],$$
$$\dot{\theta} = -\alpha(U_n - U_s)\sin\theta/R, \qquad (2)$$
$$\dot{dl}/dl = \mathring{R}/R.$$

This leads by a straightforward analysis to

$$\dot{\lambda}_{nc} = \alpha\left[\frac{\beta}{R} - (U_n - U_s)\cos\theta\right]\frac{\partial\lambda}{\partial R} + \alpha(U_n - U_s)\frac{\sin\theta}{R}\frac{\partial\lambda}{\partial\theta} + \alpha(U_n - U_s)\frac{\cos\theta}{R}\lambda. \qquad (3)$$

In (2) and (3), $\beta = (\kappa/4\pi)\ln(R/a)$ and $\alpha = B\rho_n/2\rho$, where B is a friction parameter that has been determined from independent experiments on rotating helium.[8]

Although the local self-induced velocity given by Eq. (1) will not in itself affect a homogeneous distribution, $\vec{v}_1$ itself will change in time due to the action of the higher order derivatives $\vec{r}'''$, $\vec{r}^{\,\mathrm{iv}}$, ... Repeated differentiation of Eq. (1) with respect to s yields a series of coupled equations[9]

$$\partial\vec{r}'/\partial t = \beta\vec{r}' \times \vec{r}'',$$
$$\partial\vec{r}''/\partial t = \beta\vec{r}'' \times \vec{r}''' + \beta\vec{r}' \times \vec{r}^{\,\mathrm{iv}}, \qquad (4)$$
$$\partial\vec{r}'''/\partial t = 2\beta\vec{r}'' \times \vec{r}^{\,\mathrm{iv}} + \beta\vec{r}' \times \vec{r}^{\,\mathrm{v}},$$

and so on. Differentiation of Eq. (1) with respect to time then gives

$$\partial\vec{v}_1/\partial t = \beta\vec{v}_1 \times \vec{r}''' + \beta^2\vec{r}' \times (\vec{r}' \times \vec{r}^{\,\mathrm{iv}}). \qquad (5)$$

Thus $\vec{v}_1$ changes because of $\vec{r}'''$ and $\vec{r}^{\,\mathrm{iv}}$, and these themselves are changing due to the action of still higher-order derivatives. To truncate this infinite coupled chain, we make the following statistical assumptions: (a) the distributions of $\vec{r}'''$, $\vec{r}^{\,\mathrm{iv}}$, ... do not have any significant directional preferences, and (b) *as one moves along the line, $\vec{r}'', \vec{r}''', \vec{r}^{\,\mathrm{iv}}$, ... become randomized in a characteristic distance δ equal to the interline distance $L^{-1/2}$.* Assumption (b) may be viewed as an assumption of maximum smoothness, since the random interline velocity fields will certainly produce randomization over a distance δ. Although (b) in no way restricts the scale of r'', it immediately implies the order-of-magnitude relations $r''' \sim \bar{r}''/\delta$, $r^{\mathrm{iv}} \sim \bar{r}''/\delta^2$, etc., where $\bar{r}'' = \langle R^{-1}\rangle$ is the average curvature in the vortex tangle. In addition, it is obvious from Eqs. (4) that the characteristic *time* over which $\vec{r}'', \vec{r}''', \vec{r}^{\,\mathrm{iv}}$, ... will randomize is $\tau \sim \delta/\bar{v}_1$, as one might expect. The quantities δ, $\bar{r}''$, and $\bar{v}_1$ must of course be determined self-consistently from the equation of motion.

It now follows from Eq. (5) that the third derivative term gives rise to a random walk in the direction of $\vec{v}_1$ with a typical angular displacement of order $\beta(\bar{r}''/\delta)(\delta/\bar{v}_1) \approx 1$ in a characteristic time τ. This leads directly to an angular diffusion contribution

$$\dot{\lambda}_3 = \frac{\bar{v}_1}{4\delta}\left(\frac{\partial^2\lambda}{\partial\theta^2} + \mathrm{ctn}\theta\,\frac{\partial\lambda}{\partial\theta}\right). \qquad (6)$$

The fourth derivative term gives rise to steps $\Delta\vec{r}'' \sim \beta(\bar{r}''/\delta^2)(\delta/\bar{v}_1)\vec{a} \approx \vec{a}/\delta$, where $\vec{a}$ is a random vector of order 1. This has two distinct consequences. Most importantly, one notes that $r_{\mathrm{new}}'' = |\vec{r}'' + \vec{a}/\delta| \sim (r''^2 + \delta^{-2})^{1/2}$ when averaged over $\vec{a}$ implies a monotonic increase in curvature. We will show later that $\bar{r}'' \sim \delta^{-1}$, so that this effect can be represented approximately by $\dot{R} = -\bar{v}_1 R^2/\delta^2$. It reflects nothing more than the fact that large random changes in the vector curvature are much more likely to kink up a straight line than to straighten out a highly curved line. The contribution made by this kinking term is

$$\dot{\lambda}_{4,k} = \frac{\bar{v}_1}{\delta^2}\left(R^2\frac{\partial\lambda}{\partial R} + 4R\lambda\right). \qquad (7)$$

The random steps in $\vec{r}''$ also imply that the distribution will relax diffusively in R-θ space over a characteristic distance $R^2/(R+\delta)$ in the characteristic time τ. From the standard formula $\langle\Delta x^2\rangle = 6D\tau$, one then estimates a crude diffusion contribution

$$\dot{\lambda}_{4,d} = \frac{\bar{v}_1}{6\delta}\left(\frac{R}{R+\delta}\right)^2 \times \left(R^2\frac{\partial^2\lambda}{\partial R^2} + 2R\frac{\partial\lambda}{\partial R} + \frac{\partial^2\lambda}{\partial\theta^2} + \mathrm{ctn}\theta\,\frac{\partial\lambda}{\partial\theta}\right). \qquad (8)$$

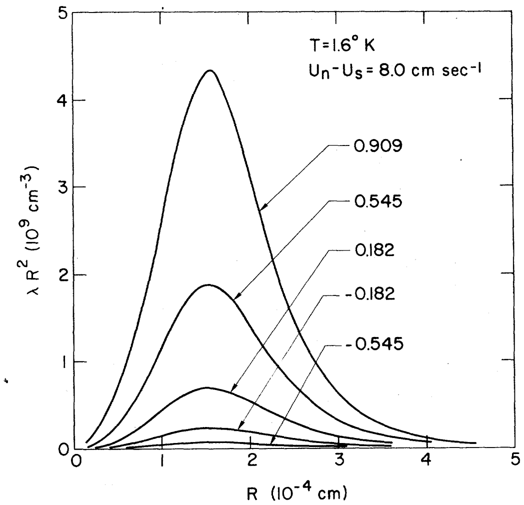

FIG. 1. Typical equilibrium distribution. Curves are drawn for various values of $\cos\theta$. Note strong peaking in $\vec{U}_n$ ($\cos\theta = 1$) direction.

The preceding argument clearly has many heuristic features. Nevertheless, it appears that the equation

$$\partial\lambda/\partial t = \dot{\lambda}_{nc} + \dot{\lambda}_3 + \dot{\lambda}_{4,k} + \dot{\lambda}_{4,d} \qquad (9)$$

which contains no adjustable parameters, provides a satisfactory rudimentary description of the vortex tangle dynamics in counterflow. Other

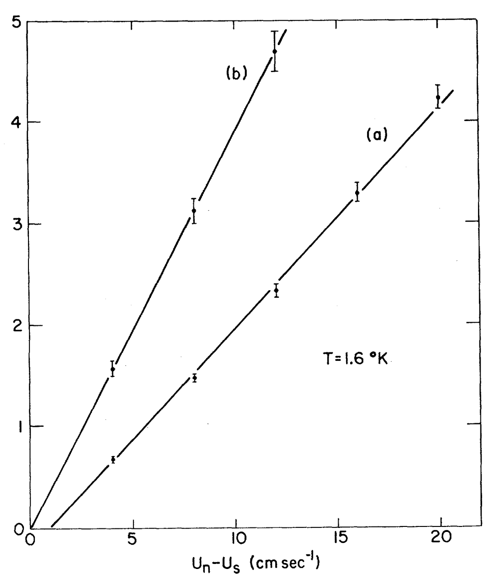

FIG. 2. Curve a shows straight-line fit to calculated values of $L^{1/2}$, plotted in units of 1000 cm^{-1}. Curve b shows straight-line fit to calculated values of $\bar{v}_{1z}$, plotted in units of cm sec^{-1}. Error bars represent numerical uncertainties arising from coarseness of the R-θ grid used in the calculations.

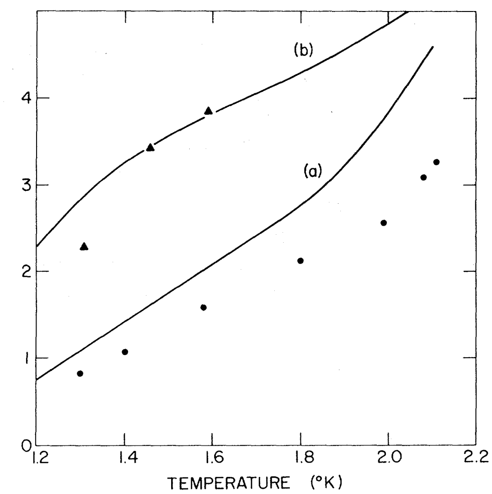

FIG. 3. Curve a gives calculated values of $A(T)$ in units of 1000 cm^{-2} sec^{-2}; the dots are the data of Ref. 1. Curve b gives calculated values of $B(T)$; the triangles are the data of Ref. 5. It is not clear how accurate the data are, but all uncertainties are at least 5%.

contributions to $\partial\lambda/\partial t$ are found to be small. In particular, the random interline velocity $\bar{v}_r \sim \kappa/2\pi\delta$ gives rise to random steps in r'' of order $(\kappa/2\pi\delta^3)\tau$. This leads to terms identical in form to $\dot{\lambda}_{4,k}$ and $\dot{\lambda}_{4,d}$, but with $\bar{v}_1$ replaced by $\bar{v}_r$. Calculations, however, always give the result that $\bar{v}_1 \gg \bar{v}_r$, showing that $\bar{v}_1$ *and not* $\bar{v}_r$ *is the dominant characteristic velocity*. In further contradiction to previous assertions, line-line crossing events do not appear to represent a major decay mechanism. On the other hand, the annihilation mechanisms $\dot{\lambda}_3$, $\dot{\lambda}_{4,k}$, and $\dot{\lambda}_{4,d}$ have not been considered in earlier work.

Because Eq. (9) is quite complicated, it is a nontrivial exercise in numerical analysis to study its properties. So far we have found that a forward integration in time, starting from an arbitrary distribution, leads inevitably to the same steady-state distribution, depending only on $U_n - U_s$ and T. As the example in Fig. 1 shows, these distributions are highly anisotropic and are strongly peaked at $R \lesssim \delta$, in agreement with the conclusion drawn by Ashton and Northby,[5] and consistent with the assumptions we have made in deriving our theory. Figure 2 shows the calculated variation with $U_n - U_s$ of $L^{1/2}$ and of the mean vortex tangle drift velocity $\bar{v}_{1z}$. Vinen[1] found experimentally that $L^{1/2} = A(T)(|U_n - U_s| - v_0)$, where $v_0 \sim 1$ cm sec^{-1}, and Ashton and Northby[5] observed that $\bar{v}_{1z} = B(T)(U_n - U_s)$. The predictions of the theory are seen to be in excellent agreement with both sets of observations, including even the presence of the small parameter v_0. Perhaps even more surprising is the numerical agreement[10] between the predicted and the observed values of $A(T)$ and $B(T)$, seen in Fig. 3. One may conclude that Eq. (9), which is very different in spirit from previous work on the subject, is remarkably successful in predicting the steady-state properties of superfluid turbulence. Further studies of this equation, in relation to such topics as critical velocities and transient effects, are currently under way.

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[9]The nonlocal terms $O(1)$ are neglected, and β is treated as a constant. These approximations involve errors of order 10%.

[10]Both the experimental and theoretical points above 2°K have been calculated assuming $B = 0.85$. There is considerable uncertainty about how to interpret the data which determine L at these temperatures, so that deviations seen there are not necessarily real.

Dislocation Inertial Effects in the Plastic Deformation of Dilute Alloys of Lead and Copper*

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Strong experimental evidence is obtained that the maximum observed in the temperature dependence of the flow stress of copper alloys is a dislocation inertial effect. By using the superconducting effect in lead as a key, it is found that the maximum occurs when dislocations become critically damped.

A maximum in the temperature dependence of the flow stress is found in copper alloyed with zinc,[1] silver,[2,3] nickel,[4,5] germanium,[5] aluminum,[3] and silicon.[3] A similar effect is also found in lead,[6] silver,[3,7] and gold.[3] This result is unexpected on the basis of existing theories of flow stress,[8] based on a quasistatic-rate-theory process, which predict a monotonically decreasing flow stress with temperature. It was already suggested by Suzuki and Ishii,[4] and by Kamada and Yoshizawa,[5] that the effect may arise from a dynamic overshooting of barriers opposing dislocation motion. Independently, from an analysis of the temperature dependence of flow-stress measurements in superconductors, Granato[9] predicted the existence of such a maximum. We give here strong experimental evidence that at low temperatures the flow stress of both superconductors and normal metals is determined by the dynamic behavior of dislocations. This is done by using internal friction measurements instead of macroscopic flow stress measurements and by comparing results for copper and lead with two different impurity concentrations.

The strain rate of a crystal which contains a density Λ of dislocations of Burgers vector b, moving at the average velocity v is

$$\dot{\epsilon} = \Lambda b v. \tag{1}$$

In the traditional theories of plasticity[8] it is supposed that the rate-limiting step is provided by the overcoming of the obstacles by thermal fluctuations. It is implicitly assumed that the process is a quasistatic one in which the dislocations do not overshoot the barriers by reason of their inertia. The average dislocation velocity is then given by

$$v = d\nu \exp[-H(\sigma)/kT], \tag{2}$$

where d is the average displacement per thermally activated event, ν is an effective attack frequency,[10] $H(\sigma)$ is the free enthalpy of activation required to overcome the obstacle, σ is the applied stress, and kT has its usual meaning.