PHYSICAL REVIEW LETTERS

VOLUME 38 | 10 JANUARY 1977 | NUMBER 2

Lagrangian for Diffusion in Curved Phase Space

Robert Graham

Universität Essen–Gesamthochschule, Fachbereich Physik, Essen, West Germany

(Received 7 October 1976)

A Lagrangian is obtained by deriving the path-integral representation of the diffusion process. It can be applied, e.g., to nonequilibrium thermodynamics and to quantized motion in general relativity. In second quantization it is shown to lead to a particularly well-behaved energy-momentum tensor as a source of gravity.

The recent development of systematic approximation schemes for classical processes[1] has generated conisderable interest in obtaining a path-integral solution for the conditional probability density

$$P(q,t;q_0,t_0)=\int_{q(t_0)=q_0}^{q(t)=q} D\mu[q(\tau)]\exp[-\int_{t_0}^{t} d\tau L(\dot{q}(\tau),q(\tau))] \tag{1}$$

of a nonlinear continuous Markov process $[q(t)=q_1(t),q_2(t),\ldots,q_n(t)]$. Such a process is described by the Fokker-Planck equation

$$\frac{\partial P}{\partial t}=\underline{L}P,$$

$$\underline{L}=-\tfrac{1}{2}\underline{p}_\nu\underline{p}_\mu Q_{\nu\mu}(q)-i\underline{p}_\nu K_\nu(q)+V(q), \tag{2}$$

where $\underline{p}_\nu\equiv -i\partial V(q)/\partial q_\nu$ is zero for stochastic processes. Since Eq. (2) gives a Euclidean representation of Schrödinger's equation, the path integral (1) also has well-known applications there and $V(q)\neq 0$ is of interest in that case.

Procedures for the derivation of Eq. (1) have been proposed recently[2] which seem to indicate that a unique representation (1) does not exist. In a separate paper I will show that these ambiguities are not real in the sense that they arise from an inadequate treatment of various limits. Here I would like to indicate the derivation and discuss the result for the Lagrangian $L[\dot{q}(\tau),q(\tau)]$. This Lagrangian coincides with the one derived earlier[3,4] for $Q_{\nu\mu}(q)=Q_{\nu\mu}$ independent of q, which has found important applications in critical dynamics in the meantime.[5] For the case with $Q_{\nu\mu}$ dependent on q, my result is novel. It rectifies an earlier incorrect result of mine[3] obtained by an oversimplified discretization method. For one-dimensional phase space (which cannot have curvature) a recent result of Horsthemke and Bach[6] is reobtained, while a result due to Stratonovich[7] for multidimensional processes is found to hold only for flat phase spaces [the metric which we use is defined after Eq. (11)]. For curved phase spaces the Lagrangian seems not to have been known earlier. After giving its derivation, I will briefly discuss an application to general relativity below. Applications to nonequilibrium thermodynamics will be considered elsewhere.

The derivation of Eq. (1) from Eq. (2) is carried out essentially in two steps. First, the ordered operator function $\underline{\tilde{S}}(\underline{\tilde{p}}/\sqrt{\epsilon}, q, t_0+\epsilon, t_0)$ defined by

$$N\{\tilde{S}(\underline{p}, q, t, t_0)\} \equiv \exp\underline{L}(t-t_0) \tag{3}$$

(where N is an ordering operator, which brings all $\underline{\tilde{p}} \equiv \sqrt{\epsilon}\,\underline{p}$ to the left of all q's regardless of their nonzero commutator) is determined as a power-series expansion in $\sqrt{\epsilon}$ up to terms of order ϵ as an ordered function of $\underline{\tilde{p}}$ and q. The function $\tilde{S}$ is closely related to the Green's function $P(q_1, t_1; q_0, t_0)$ of Eq. (2) since

$$P(q_1, t_1; q_0, t_0) = \int \frac{dp}{(2\pi)^n} \exp[ip(q_1 - q_0)]\, \tilde{S}(p, q_0, t, t_0), \tag{4}$$

where p are n real c-number variables. From Eq. (4) and the property,

$$P(q_N, t_N; q_0, t_0) = \int \left\{ \prod_{j=1}^{N-1} dq_j\, P(q_{j+1}, t_{j+1}; q_j, t_j) \right\} P(q_1, t_1; q_0, t_0), \tag{5}$$

I obtain (with $\epsilon = t_{j+1} - t_j$)

$$P(q_N, t_N; q_0, t_0) = \lim_{\epsilon\to 0} \int \prod_{j=1}^{N-1} [dq_j] \exp \sum_{k=0}^{N-1} \ln \int \frac{d\tilde{p}}{(2\pi\sqrt{\epsilon})^n} \exp\left[\frac{i\tilde{p}(q_{k+1} - q_k)}{\sqrt{\epsilon}}\right] \tilde{S}\left(\frac{\tilde{p}}{\sqrt{\epsilon}}, q_k, \tau+\epsilon, \tau\right). \tag{6}$$

The second step of my derivation consists in the evaluation of the right-hand side of Eq. (6). Since I will require the path integral (1) to take the same form for continuous nondifferentiable functions $q(\tau)$ as for continuous differentiable ones (since the latter may be considered as a special case of the former), it is sufficient to consider the right-hand side of Eq. (6) just for continuous differentiable functions $q(\tau)$. In that case I may put $(q_{k+1} - q_k)/\epsilon \to \dot{q}(\tau)$. (Note that the discretization process in the reverse direction is not unique in the same way.) Let us separate the logarithm in Eq. (6) in the limit $\epsilon \to 0$ into a regular part $\Omega[\epsilon, \dot{q}(\tau), q(\tau)]$ vanishing at least like ϵ, and a remaining irregular part $\mathrm{Ir}[\epsilon, \dot{q}(\tau), q(\tau)]$. Then Eq. (1) is obtained with

$$D\mu[q(\tau)] = \lim_{\epsilon\to 0} \left\{ \prod_{j=1}^{N-1} [dq_j] \exp \mathrm{Ir}[\epsilon, \dot{q}(\tau), q(\tau)] \right\}, \tag{7}$$

$$L[\dot{q}(\tau), q(\tau)] = -\lim_{\epsilon\to 0} \epsilon^{-1}\Omega[\epsilon, \dot{q}(\tau), q(\tau)]. \tag{8}$$

Applying this method to Eq. (2) I obtain

$$D\mu[q(\tau)] = \lim_{\epsilon\to 0} \left[\left(\prod_{j=1}^{N-1} \frac{dq_j}{[Q(q_j)(2\pi\epsilon)^n]^{1/2}} \right) \frac{1}{[Q(q)(2\pi\epsilon)]^{1/2}} \right], \tag{9}$$

$$L[\dot{q}(\tau), q(\tau)] = \tfrac{1}{2} Q_{\nu\mu}{}^{-1}(\dot{q}_\nu - h_\nu)(\dot{q}_\mu - h_\mu) + \tfrac{1}{2}\sqrt{Q}\, \frac{\partial}{\partial q_\nu}\left(\frac{h_\nu}{\sqrt{Q}}\right) - V(q) + \tfrac{1}{12} R, \tag{10}$$

$$h_\nu = K_\nu - \tfrac{1}{2}\sqrt{Q}\, \frac{\partial}{\partial q_\mu}\left(\frac{Q_{\nu\mu}}{\sqrt{Q}}\right), \tag{11}$$

and $Q = \|Q_{\nu\mu}\|$. R is a complicated algebraic form in $Q_{\nu\mu}$ and its first- and second-order derivatives with respect to q. Equation (10) is made beautiful by the fact that R is just the Riemann curvature scalar[8] if $Q_{\nu\mu}{}^{-1}$ is defined as the covariant metric tensor in phase space. It is easily shown that $\dot{q}_\nu$ and h_ν then transform like contravariant vectors, whereas the Lagrangian (10) and the measure (9) transform like a scalar and a scalar density of weight -1, respectively. Equation (10) for $V \equiv 0$ gives the most general Lagrangian needed in nonequilibrium thermodynamics [a formulation in terms of space integrals is a trivial generalization of Eq. (10)]. The various special cases contained in Eq. (10) for $R = 0$ have already been discussed above. If the time is taken imaginary and again $R = 0$, various well-known results of Feynman's space-time approach to quantum mechanics (including the case of an external electromagnetic field) are contained in Eqs. (9) and (10).[9]

For $R \neq 0$, a rather amusing application of my result to general relativity can be given. For this purpose, it must be recognized that Eq. (2), after the replacement $t \to -iu$ (where u is a real, redundant, fifth parameter), is a possible (although somewhat unusual) representation of the Klein-Gordon or Dirac equation.[10] Let us first assume that $R=0$ (i.e., no gravitational fields are present) and look at the motion of a free scalar Klein-Gordon particle with real wave function φ and mass μ_0. In that case, Eq. (2) [with $t\equiv -iu$, $q_\mu \equiv x^\mu$, $V=\mu_0^2/2$, $h_\nu \equiv 0$, $(-\sqrt{Q})P\equiv\varphi$, and $Q_{\mu\nu}^{-1} = g_{\mu\nu}$ as metric tensor with the signature (+, −, −, −) in flat space] gives an adequate description in first quantization, if only solutions independent of the redundant fifth parameter u are singled out at the end.[10] The "classical" Lagrangian for a single particle in this representation can then immediately be read off Eq. (10) so that

$$L=-\tfrac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu-\tfrac{1}{2}\mu_0^2, \tag{12}$$

where the dots are now derivatives with respect to u. The principle of general covariance implies that Eq. (12) also holds in gravitational fields where $R\neq 0$. However, this is compatible with Eq. (10) if and only if I change $V=\mu_0^2/2$ in Eq. (2) into $V=\mu_0^2/2+R/12$. Thus, in a gravitational field the Klein-Gordon equation reads

$$g^{\mu\nu}\partial_\mu\partial_\nu\varphi=-\mu_0^2\varphi-\tfrac{1}{6}R\varphi, \tag{13}$$

where ∂_μ denotes a covariant derivative. Note that φ, in contrast to a classical trajectory, always probes a finite region in space-time. Hence, the direct application of the principle of equivalence or general covariance to Eq. (13) is not possible, which is why I first considered the "classical" Lagrangian (12), where the application is possible. Equation (13) has the Lagrangian density

$$\mathcal{L}=\frac{1}{2}g^{\mu\nu}\frac{\partial\varphi}{\partial x^\mu}\frac{\partial\varphi}{\partial x^\nu}-\frac{1}{2}\mu_0^2\varphi^2-\frac{1}{12}R\varphi^2+\frac{1}{16\pi G}R, \tag{14}$$

where the Lagrangian density of the gravitational field has been added with the last term in Eq. (14).[11] As a consequence of the new term, $-\frac{1}{12}R\varphi^2$, the gravitational field equations, obtained from the action principle, are changed in such a way that the source of gravity no longer is the conventional energy-momentum tensor $T_{\mu\nu}$. In place of $T_{\mu\nu}$, a new tensor

$$\theta_{\mu\nu}=T_{\mu\nu}-\tfrac{1}{6}(\partial_\mu\partial_\nu-g_{\mu\nu}\partial^\lambda\partial_\lambda)\varphi^2 \tag{15}$$

appears to lowest order in the gravitational coupling. $\theta_{\mu\nu}$ still has all desirable properties of an energy-momentum tensor.[12] Callan, Coleman, and Jackiw[12] have shown that $\theta_{\mu\nu}$, in contrast to $T_{\mu\nu}$, has finite matrix elements in a renormalizable field theory (e.g., φ^4 theory) to all orders of perturbation theory in any coupling constants (except the gravitational coupling, which leads to a nonrenormalizable theory). While these authors had to change the field equations in second quantization in an *ad hoc* manner in order to introduce $\theta_{\mu\nu}$ in place of $T_{\mu\nu}$, my result (10) explains how this new energy-momentum tensor appears in a natural way as a consequence of the principle of general covariance for the classical trajectories in space-time. By way of an example I have thus shown something very surprising and remarkable: Some of the divergencies of quantized field theory can be avoided by the judicious use of the equivalence principle.

I would like to thank Werner Horsthemke for a useful and stimulating discussion and for drawing my attention to Ref. 7. I also thank P. Hänggi and H. Dekker for sending preprints prior to publication. Konny Kaufmann's enthusiasm and Fritz Haake's useful suggestions are gratefully acknowledged.

[1] P. C. Martin, E. D. Siggia, and H. A. Rose, Phys. Rev. A 8, 423 (1973); H. K. Janssen, to be published.

[2] H. Haken, Z. Phys. B 24, 321 (1976); P. Hänggi and S. J. Nettel, to be published.

[3] R. Graham, in *Quantum Statistics in Optics and Solid State Physics* (Springer, Berlin, 1973).

[4] R. Graham, in *Fluctuations, Instabilities and Phase Transitions*, edited by T. Riste (Plenum, New York, 1976).

[5] R. Bausch, H. K. Janssen, and H. Wagner, to be published.

[6] W. Horsthemke and A. Bach, Z. Phys. B 22, 189 (1975).

[7] R. L. Stratonovich, in *Selected Translations in Mathematical Statistics and Probability* (American Mathematical Society, Providence, R. I., 1971), Vol. 10, p. 273.

[8] Cf. S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972), pp. 141–142.

[9] Cf. R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals*, (McGraw-Hill, New York, 1965); M. Kac, *Probability and Related Topics, Physical Sciences* (Interscience, New York, 1965).

[10] R. P. Feynman, Phys. Rev. 80, 440 (1950), Appendix A, and Phys. Rev. 84, 108 (1951), Appendix D.

[11] The notational conventions of C. G. Callan, Jr., S. Coleman, and R. Jackiw, Ann. Phys. (N.Y.) 59, 42 (1970), Sect. 6 are used.

[12] Callan, Coleman, and Jackiw, Ref. 11.