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Deviations from Dynamic Scaling in Helium and Antiferromagnets

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We compute up to order ϵ^2 all dynamics transients (subleading exponents) for helium and symmetric antiferromagnets. We also discuss the relevance for helium of a fixed point which leads to a "weak" dynamic scaling.

Critical dynamics of systems like helium and symmetric antiferromagnets (AF) involving reversible mode coupling has been widely studied[1,2] using Wilson's[3] recursion method on stochastic Langevin-like equations or techniques of field renormalization which are useful to demonstrate scaling properties and get a relatively easy access to higher-order computations.[4,5] We have taken advantage of these techniques to investigate the two-loop (i.e., second order in $\epsilon = 4-d$) approximations for the critical dynamic behavior of helium and $O(n)$ symmetric antiferromagnets. In particular we obtain to order ϵ^2 all subleading exponents (transients) which govern corrections to dynamic scaling. This allows us to distinguish in the ϵ-n plane, besides region (I) where standard dynamic scaling[6] holds, a region (II) involving "weak" dynamic scaling due to the occurrence of a "dangerous" irrelevant parameter.[7] In (I) the dynamic exponent is[6] $z=d/2$; in (II) it turns out to be $z=\frac{1}{2}(d+\omega_\lambda)$, where ω_λ is one of the three (He) or two (AF) subleading exponents which are computed here. The possible, but unlikely, physical relevance of region II for He is also discussed.

We write the Langevin stochastic equations for He introduced by Halperin, Hohenberg, and Siggia[1] in the form

$$\dot{\psi}_0{}^\alpha(t) = -\Gamma_0(1+ib_0)\frac{\delta\mathcal{H}}{\delta\psi_0{}^{\alpha *}(t)} + ig_0\psi_0{}^\alpha(t)\frac{\delta\mathcal{H}}{\delta E_0(t)} + \eta^\alpha(t), \tag{1}$$

$$\dot{E}_0(t) = \Lambda_0\nabla^2\frac{\delta\mathcal{H}}{\delta E_0(t)} + ig_0\sum_\alpha\left(\psi_0{}^{\alpha *}(t)\frac{\delta\mathcal{H}}{\delta\psi_0{}^{\alpha *}(t)} - \psi_0{}^\alpha(t)\frac{\delta\mathcal{H}}{\delta\psi_0{}^\alpha(t)}\right) + \zeta(t), \tag{2}$$

where the space argument is understood and

$$\mathcal{H} = \int d^dx\{\textstyle\sum_\alpha(|\nabla\psi_0{}^\alpha|^2 + r_0|\psi_0{}^\alpha|^2) + \frac{1}{2}E_0{}^2 + \gamma_0E_0\sum_\alpha|\psi_0{}^\alpha|^2 + (u_0/3!)(\sum_\alpha|\psi_0{}^\alpha|^2)^2\}. \tag{3}$$

Here $\psi_0{}^\alpha$ is the (complex) order parameter field ($\alpha=1$ for He; we use $\alpha = 1, \ldots, n/2$), E_0 is the conserved "energy," and η^α and ζ are the associated Gaussian noises whose correlations are related to the kinetic coefficients Γ_0 and $-\Lambda_0\nabla^2$ by Einstein relations. γ_0 and u_0 are the static couplings and g_0 is the reversible mode coupling constant. Equations for the AF case, where the order parameter ψ_0 is a real n-component vector ($n=3$) and $E_0{}^{\alpha\beta}$ represent the generators of the $O(n)$ symmetry, are simply related[1,2] to (1)–(3).

These equations are cast in a form tractable by field theoretic methods by using the Martin-Siggia-Rose Lagrangian[8] which involves conjugate variables $\hat{\psi}_0{}^\alpha$ and $\hat{E}_0$. The *statics* as contained in

Eq. (3) has already been discussed,[9] and is renormalized[4,5] through

$$\psi_0 = Z_\psi^{-1/2}\psi, \quad E_0 = Z_E^{-1/2}E, \tag{4}$$

$$u_0 = u^\epsilon u Z_u/Z_\psi^2, \quad \gamma_0 = \mu^{\epsilon/2}\gamma Z_\gamma/Z. \tag{5}$$

Static asymptotic behavior is dominated by the leading exponents

$$\eta_\Gamma \equiv \eta = \mu d\ln Z_\psi/d\mu, \quad \eta_E = \mu d \ln Z_E/d\mu,$$

taken at fixed point values u^* and γ^*. (The derivatives are meant at fixed bare parameters, μ being a wave vector characterizing the so-called subtraction point.) The fixed point is the stable zero of the Wilson functions $W_j \equiv \mu\, dj/d\mu$ $(j=\gamma, u)$, and slopes $\omega_j \equiv dW_j/dj$ taken at the fixed point are the subleading exponents (transients), with $\omega_\gamma = -\alpha/\nu$ and $\gamma^*=0$ for He.

The *dynamic* field theory is more subtle to cast in renormalized form. Indeed the basic Green's functions constructed with ψ, $\hat{\psi}$, E, and $\hat{E}$ are not simply related to the static ones; in particular their zero-frequency limit is *not* a static function. On the other hand, the renormalization factors Z_l $(l=\psi, E, u, \gamma$ for the statics) are usually fixed in a standard way by imposing normalization conditions on these functions at some arbitrary (subtraction) point in wave-vector–frequency space. This procedure, therefore, inevitably mixes statics and dynamics in the renormalized perturbation expansion, making the above defined static functions η_j and W_j dependent upon purely dynamic parameters. This leads of course to the same physical results (e.g., exponents) but in a both unphysical and complicated fashion. The way out is either to impose normalization conditions with good static limits (using response functions rather than basic Green's functions) or to resort to the so-called minimal renormalization procedure[10] which does not require an explicit choice for the normalization conditions and separates static and dynamic renormalization. We use this last procedure. Together with a suitable renormalization for $\hat{\psi}$ and $\hat{E}$, we need to introduce

$$\Gamma_0 = \Gamma Z_\Gamma/Z_\psi, \tag{6}$$

$$\Lambda_0/\Gamma_0 = (\Lambda/\Gamma)\, Z_E Z_\Gamma/Z_\Lambda Z_\psi, \tag{7}$$

$$b_0 = bZ_b, \tag{8}$$

$$g_0^2 = \mu^\epsilon g^2 Z_E, \tag{9}$$

with the definitions $\lambda \equiv \Lambda/\Gamma$ and $f \equiv g^2/\lambda\Gamma^2$. By differentiating at fixed bare parameters, we obtain the Wilson functions

$$W_\lambda = -\lambda(\eta_\Gamma - \eta_\Lambda), \tag{10}$$

$$W_f = -f(\epsilon + \eta_\Gamma + \eta_\Lambda + \eta_E), \tag{11}$$

respectively, from (7) and (9) which derives from a Ward identity.[1] Notice that (10) and (11) lead to $\eta_\Gamma^* = \eta_\Lambda^* = -\eta_E^* - \epsilon/2$, as soon as $\lambda^*, f^* \neq 0, \infty$.

For the cases of interest (He, symmetric AF) the relevant fixed point is at $\gamma^*=0$, $b^*=0$. We get, for the extended He case,

$$\eta_\Gamma \equiv \mu\frac{d\ln(Z_\Gamma/Z_\psi)}{d\mu} = -\frac{f\lambda}{1+\lambda} + \frac{f^2\lambda^2}{8(1+\lambda)^3}\left[4(2+\lambda)\ln\frac{(1+\lambda)^2}{\lambda(2+\lambda)} + 9(1+\lambda)(4+n)\ln\tfrac{4}{3} - (4+n) - \lambda(8+n)\right] + \frac{n+2}{2(n+8)^2}u^2(6\ln\tfrac{4}{3}-1), \tag{12}$$

$$\eta_\Lambda \equiv \mu\frac{d\ln(Z/Z_E)}{d\mu} = -\frac{fn}{4} + \frac{f^2 n}{8(1+\lambda)}\left[\frac{1+2\lambda}{\lambda^2}\ln\frac{(1+\lambda)^2}{1+2\lambda} - 1 - \frac{\lambda}{2}\right], \tag{13}$$

and for the AF case

$$\eta_\Gamma = -\frac{f\lambda}{1+\lambda}(n-1) + \frac{f^2\lambda^2(n-1)}{2(1+\lambda)^3}\left[(2-n+\lambda)(2+\lambda)\frac{1}{\lambda}\ln\frac{2(1+\lambda)}{2+\lambda} + (n+\lambda)\ln\frac{1+\lambda}{2\lambda} + \frac{1+\lambda}{2}(27\ln\tfrac{4}{3}-5) + n - 1\right] + \frac{n+2}{2(n+8)^2}u^2(6\ln\tfrac{4}{3}-1), \tag{14}$$

$$\eta_\Lambda = -\frac{f}{2} - \frac{f^2}{4(1+\lambda)}\left[\frac{[\lambda(n-2)-1](2\lambda+1)}{\lambda^2}\ln\frac{(1+\lambda)^2}{1+2\lambda} + (3-n)\frac{\lambda}{2} + 1\right]. \tag{15}$$

The transients are the eigenvalues ω_j of the 5×5 matrix $\partial W_j/j, l = u, \gamma, f, b$. With the procedure used here, the static (u,γ) and dynamic (f, λ, b) parts of the matrix decouple. For symmetric systems $(\gamma^* = 0)$ only the f and λ components remain coupled. Solving for fixed points given as the zeros of (10) and (11), we find, besides the fixed point I of Halperin, Hohenberg, and Siggia, a fixed point II with $\lambda^* = \infty$.

The transient exponents for fixed point I are

$$\omega_f = \epsilon - 0.3135\epsilon^2 = (0.69; 0.78),$$
$$\omega_\lambda = \tfrac{1}{4}\epsilon - 0.1498\epsilon^2 = (0.10; 0.17), \quad (16)$$
$$\omega_b = \tfrac{3}{4}\epsilon - 0.3618\epsilon^2 = (0.39; 0.53),$$

for He; and

$$\omega_f = \epsilon - 0.3643\epsilon^2 = (0.64; 0.76),$$
$$\omega_\lambda = \tfrac{3}{8}\epsilon - 0.1156\epsilon^2 = (0.26; 0.29), \quad (17)$$

for AF, where the numbers in parentheses are the value for $\epsilon = 1$ followed by the Padé-Borel[11] value. For fixed point II they are

$$\omega_f = \epsilon - 0.2374\epsilon^2 = (0.76; 0.82),$$
$$\omega_\lambda = -\tfrac{1}{3}\epsilon + 0.2887\epsilon^2 = (-0.04; -0.20), \quad (18)$$
$$\omega_b = \tfrac{2}{3}\epsilon - 0.1959\epsilon^2 = (0.47; 0.53),$$

for He; and

$$\omega_f = \epsilon - 0.2064\epsilon^2 = (0.79; 0.84),$$
$$\omega_\lambda = -\tfrac{3}{5}\epsilon + 0.0946\epsilon^2 = (-0.51; -0.49), \quad (19)$$

for AF. Gunton and Kawasaki[2] had already noticed this fixed point for AF, where it shows a definite instability. However for He we get a sufficiently small $\omega_\lambda^{\mathrm{II}}$ for $\epsilon = 1$ to allow for inquiring into a possible stability of II. Indeed in the ϵ-n plane, the line $\omega_\lambda = 0$ which separates regions I and II is tangent, for $\epsilon \simeq 0$, to T_2:$n = 4 - p\epsilon$, $p = 19 \times \ln(\frac{4}{3}) - \frac{8}{3} = 2.80$, and the physical point $\epsilon = 1, n = 2$ is on the II side of T_2. Which side it lies of $\omega_\lambda = 0$ cannot be decided for sure to this order of the expansion. The situation bears some analogy with that of the line $\omega_\gamma = 0$ which separates the symmetric and asymmetric regions and is tangent to T_1:$n = 4 - 4\epsilon$ leaving the physical point on the symmetric side of T_1.[12] In this last case it follows from $\alpha/\nu < 0$ that $\epsilon = 1, n = 2$ does lie in the symmetric region although the ϵ expansion cannot guarantee it. The situation is clearer for AF (T_2:$n = \frac{3}{2} + 0.55\epsilon$) where the physical point ($\epsilon = 1, n = 3$) appears definitely inside region I.

The three dynamic transients computed in (16) for He [two in (17) for AF] are to be taken into account in experimental fits. The slow transient ω_λ may account for some of the discrepancies left over in fitting thermal conductivity[13] or amplitude ratios.[1,14]

In view of the small values obtained for ω_λ at both fixed points [(16) and (18)], it is legitimate to investigate what consequences would follow if fixed point II were the relevant one for He.

Dynamic scaling.—By integrating Callan-Symanzik equations, one obtains a characteristic frequency of the form

$$\omega_c(k, \xi) = k^z \Omega(k\xi, (\mu\xi)^{\omega_\lambda}), \quad (20)$$

where ξ is the correlation length and the last argument in Ω refers to the λ dependence.

Above T_c ω_ψ is regular in λ, and therefore restricted scaling holds with a modified exponent

$$z = \tfrac{1}{2}(d + \omega_\lambda), \quad (21)$$

$$\omega_\lambda = \eta_\Gamma{}^* - \eta_\Lambda{}^*, \quad (22)$$

as follows from (10) and (11). On the other hand, ω_E is asymptotically proportional to λ, yielding

$$\omega_E \sim k^{z-\omega_\lambda} \Omega_E(k\xi), \quad (23)$$

a violation of extended dynamic scaling. It follows that the thermal conductivity behaves as $\xi^{(\epsilon+\omega_\lambda)/2}$ instead of $\xi^{\epsilon/2}$. A precise measurement of the asymptotic ξ dependence would give ω_λ. If we were to accept as the asymptotic one the value suggested by Ahlers[13] ($\gtrsim 0.55$), we would get the estimate $\omega_\lambda \sim 0.1$.

Below T_c the characteristic frequency for $k\xi_- \ll 1$ takes the form $\omega_- \sim ck + iDk^2$, where the second-sound velocity c vanishes like $\xi_-^{\epsilon/2-1}$ in agreement with hydrodynamics, whereas the second-sound attenuation coefficient D goes like $\xi_-^{(\epsilon+\omega_\lambda)/2}$, a violation of dynamic scaling.

Amplitude ratios.—Some of the amplitude ratios possess a nonsmooth λ dependence for $\lambda \to \infty$. They acquire a nonuniversal character, thus leaving some room for data fitting. On the other hand they also acquire a ξ dependence for fixed $k\xi$; e.g., ω_E/ω_ψ and, below T_c, $\mathrm{Im}\,\omega_-/\mathrm{Re}\,\omega_-$ are proportional to λ and $\lambda^{1/2}$, i.e., $\xi^{\omega\lambda}$ and $\xi^{\omega\lambda/2}$, respectively. For the (last) measurable ratio this entails a 25% variation in the experimental range, not incompatible with the accoustical data of Tyson.[15] For the universal ratio $\omega_E/\mathrm{Im}\,\omega^-$, the previous (Halperin, Hohenberg, and Siggia) not too good agreement is made worse by a factor of 2. Finally the standing conflict with light-scattering experiments[16] is rather aggravated: The second-sound attenuation coefficient D, which remains constant in these experiments, varies like $\xi^{1/2}$ for fixed point I, and like $\xi^{(1+\omega_\lambda)/2}$ for fixed point II. These last two arguments are probably the best against the physical relevance of fixed point II.

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Phase Slippage without Vortex Cores: Vortex Textures in Superfluid ^{3}He

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The characteristic dissipation process for conventional superfluid flow is phase slippage: motion of quantized vortices in response to the Magnus force, which allows finite chemical potential differences to occur. Topological considerations and actual construction are used to show that in liquid ^{3}He-A, textures with vorticity but no vortex core can easily be constructed, so that dissipation of superfluid flow can occur by motion of textures alone without true vortex lines, dissipation occurring via the Cross viscosity for motions of $\hat{l}$.

Usually dissipative relaxation of the order parameter of a broken-symmetry condensed system occurs by motion of order-parameter singularities. For instance, magnetic hysteresis involves the motion of domain walls, slip of solids that of dislocations, and self-diffusion that of vacancies or interstitials. These are 2-, 1-, and 0-dimensional "order-parameter singularities." All are characterized by a "core" of atomic dimension where the order parameter departs substantially from its equilibrium value.

One of the clearest examples of this general rule is phase slippage in superconductors (flux flow and creep) and in liquid helium II: The only way in which these superfluides in bulk form can sustain a gradient of chemical potential, and thus flow dissipatively, is by the continual motion of quantized vortex lines transverse to that gradient. The controlling equation is[1]

$$\langle \mu_1 - \mu_2 \rangle = \left\langle \frac{\hbar\, d(\varphi_1 - \varphi_2)}{dt} \right\rangle = h\, \frac{dn_{\text{vortices}}}{dt}, \qquad (1)$$