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Nonlinear Schrödinger Equation for Dispersive Media

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A self-consistent three-dimensional nonlinear Schrödinger equation is derived for dispersive media such as plasmas by using the Krylov-Bogoliubov-Mitropolsky multiple space-time method. The necessary and sufficient conditions for the occurrence of modulational instability and the collapse of Langmuir as well as ion-acoustic waves are obtained. In weakly turbulent systems, collapse is not possible.

It is well known that the weakly nonlinear dispersive waves in one dimension are governed by the nonlinear Schrödinger (NS) equation which admits localized stationary solutions, namely, envelope solitons and envelope holes.[1,2] Because of the mathematical complexity, most of the analytic work reported in the literature deals with the one-dimensional NS equation. To my knowledge, the only exception to this is the work of Zakharov[3] who studied the nonlinear time evolution of spherical three-dimensional Langmuir waves. By averaging over the "fast time," he arrived at a simplified dynamic model of the plasma. The only nonlinearity that appears in his model is through the ponderomotive force which the high-frequency field exerts on the ions. According to his model, the Langmuir waves undergo a spatial collapse in a finite time. In the present work, by using the Krylov-Bogoliubov-Mitropolsky[4] multiple space-time method, I self-consistently derive and analyze the three-dimensional NS equations both for nonlinear ion-acoustic and Langmuir waves in weakly turbulent plasmas. For these, I show other nonlinear effects are more important than the ponderomotive force.

Let us first consider ion-acoustic waves. For simplicity, let us take a plasma with cold ions and isothermal electrons. Neglect the electron inertia, i.e., terms of order m/M, in the three-dimensional fluid equations and assume that the slow variations of the finite amplitude of the waves are given by

$$\frac{\partial \vec{a}}{\partial t} = \epsilon \vec{A}_1 + \epsilon^2 \vec{A}_2 + \dots, \tag{1a}$$

and

$$\frac{\partial a_i}{\partial x_j} = \epsilon B_{ij}^{(1)} + \epsilon^2 B_{ij}^{(2)} + \dots. \tag{1b}$$

I can show, following Buti[5] and Kakutani and Sugimoto,[6] that the resonant secularity to order ϵ^3 can be removed by imposing the condition

$$i\frac{\partial \varphi}{\partial \tau} + \overleftrightarrow{\mathrm{P}}:\frac{\partial^2 \varphi}{\partial \vec{\mathcal{E}}\partial \vec{\mathcal{E}}} = Q|\varphi|^2\varphi + R\varphi. \tag{2}$$

This is the nonlinear Schrödinger equation in three dimensions. The various quantities appearing in Eq. (2) are defined as

$$\vec{\mathcal{E}} = \epsilon(\vec{x} - \vec{V}_g t), \quad \tau = \epsilon^2 t, \quad \varphi = (\vec{k}\cdot\vec{a})/k,$$

$$P_{\alpha\beta} = \frac{1}{2}\frac{\partial V_{g\alpha}}{\partial k_\beta} = \frac{\omega^3}{2k^4}\left[\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}(1+3\omega^2)\right], \tag{3}$$

$$Q = \frac{\omega^5}{12k^6}(3+3k^2+k^4)^{-1} \times (3k^{10}+6k^8-6k^6-29k^4-30k^2-12), \tag{4}$$

and

$$R = \frac{\omega}{2k^2}\left[\eta_1(\omega^2-k^2) + \frac{2k^2}{\omega}(\vec{k}\cdot\vec{\gamma}_1)\right], \tag{5}$$

with η_1 and γ_1 as absolute constants which can be determined by initial conditions. ω and $\vec{k}$ appearing in Eqs. (3)–(5) are related through the dispersion relation

$$\omega^2 - k^2 + k^2\omega^2 = 0, \tag{6}$$

and $\vec{V}_g = (\omega^3/k^4)\,\vec{k}$ is the group velocity. It is interesting to note that Q, the coefficient of the nonlinear term, is the same as in the one-dimensional NS equation.[6] The R term in Eq. (2) can be removed by the simple transformation $\varphi \to \varphi e^{-iR\tau}$. So henceforth, we will consider Eq. (2) without the R term.

Let us first examine the modulational stability of Eq. (2). For this, write

$$\varphi(\vec{\mathcal{E}}, \tau) = \chi^{1/2}(\vec{\mathcal{E}}, \tau)\exp[i\sigma(\vec{\mathcal{E}}, \tau)], \tag{7}$$

where χ and σ are real functions. On separating the real and the imaginary parts of Eq. (2) and on linearizing by using the relation

$$\begin{pmatrix}\chi\\ \sigma\end{pmatrix} = \begin{pmatrix}\chi_0\\ \sigma_0\end{pmatrix} + \begin{pmatrix}\chi_1\\ \sigma_1\end{pmatrix}\exp[i(\vec{K}\cdot\vec{\mathcal{E}} - \Omega\tau)],$$

one finds that the plane waves governed by dispersion relation (6) are modulationally unstable only if

$$(\vec{P}:\vec{K}\vec{K})Q < 0 \tag{8a}$$

and

$$|(\vec{P}:\vec{K}\vec{K})Q^{-1}| < 2\chi_0. \tag{8b}$$

For longitudinal modulations, i.e., $\vec{K}\,||\,\vec{k}$, inequality (8a) can be satisfied only if $(\omega^5 Q/k^4) > 0$ which in turn is satisfied for $k > 1.47$. For transverse modulations, however, instability occurs if $(Q\omega^3/k^4) < 0$, i.e., if $k < 1.47$. In general, for oblique modulation, namely $(\vec{K}\cdot\vec{k}) = Kk\cos\theta$, one finds that for $\theta \leqslant 55°$, ion waves with $k > 1.47$ are unstable but for $\theta \geqslant 59.5°$, $0 < k < 1.47$ defines the region of instability. This shows that the ion-acoustic waves with $0 < k \leqslant 1.47$, which are modulationally stable in one dimension, become unstable in the realistic situation of three-dimensions provided $\theta \geqslant 59.5°$. Next we check to see whether Eq. (2) admits localized stationary solutions and investigate the time evolution of these nonlinear waves. For this purpose, invoke spherical symmetry, i.e., take $\varphi(\vec{\mathcal{E}}, \tau) = \varphi(\mathcal{E}, \tau)$. For the spherically symmetric case, Eq. (2) can be rewritten as

$$i\frac{\partial\varphi}{\partial\tau} + \frac{\omega^3}{k^4}\left(\frac{1}{\mathcal{E}}\frac{\partial\varphi}{\partial\mathcal{E}} - \frac{3}{2}\omega^2\frac{\partial^2\varphi}{\partial\mathcal{E}^2}\right) = Q|\varphi^2|\varphi. \tag{9}$$

It is interesting to note that in the absence of the nonlinear term, Eq. (9) has the solution

$$\varphi = b\mathcal{E}^\nu K_\nu(k\mathcal{E})\exp\left(-\frac{3i}{2}\frac{\omega^5}{k^2}\tau\right), \tag{10}$$

where $\nu = (2 + 5k^2)/6k^2$, K_ν is the Bessel function of second kind, and b is a constant.

Following Hasegawa[2] and on using Eq. (7), I can show that Eq. (9) does not permit localized stationary ($\partial\chi/\partial\tau = 0$) solutions. Since it is not possible to solve the nonlinear Eq. (9), I will try to get some physical insight into the system governed by Eq. (9) by looking into some of the integrals of motion of this equation. On making the transformations $\varphi = \rho^\mu|Q|^{-1/2}\psi$ and $\mathcal{E} = (3\omega^5/2k^4)^{1/2}\rho$ Eq. (9) reduces to

$$i\frac{\partial\psi}{\partial\tau} - \nabla_\rho^2\psi + \frac{\mu(\mu-1)}{\rho^2}\psi = \eta\rho^{2\mu}|\psi|^2\psi, \tag{11}$$

where

$$\nabla_\rho^2 = (\partial^2/\partial\rho^2 - 2/\rho), \quad \mu = [1 + (3\omega^2)^{-1}],$$

and $\eta = +1$ for $Q > 0$ and -1 for $Q < 0$. Now one can easily show that

$$I_1 = \int_0^\infty d\rho\,\rho^2|\psi|^2$$

and

$$I_2 = \int_0^\infty d\rho\,\rho^2\left[|\nabla\psi|^2 + \frac{\mu(\mu-1)}{\rho^2}|\psi|^2 - \tfrac{1}{2}\eta\rho^{2\mu}|\psi|^4\right] \tag{12}$$

are integrals of motion of Eq. (11). Moreover, one finds that

$$A = \int_0^\infty d\rho\,\rho^4|\psi|^2$$

satisfies the relation

$$\frac{1}{2}\frac{d^2A}{d\tau^2} = 12I_2 - 8\mu(\mu-1)\int_0^\infty d\rho|\psi|^2 + (3+2\mu)\eta\int_0^\infty d\rho\rho^{2(\mu+1)}|\psi|^4. \tag{13}$$

Since $\mu > 1$ for $\eta = -1$, i.e., for $Q < 0$, Eq. (13) can be integrated to give

$$A < 12I_2\tau^2 + C_1\tau + C_2, \tag{14}$$

where C_1 and C_2 are constants. Since A is positive definite, Eq. (13) would have a singularity only if

$I_2 < 0$. This singularity would mean that in finite time A becomes zero, i.e., collapse would occur. However, from Eq. (12), it is evident that for $Q < 0$, I_2 cannot be negative and hence there is no collapse for ion-acoustic waves. It must be emphasized that the necessary condition for the collapse to take place is $Q > 0$ which corresponds to modulational instability of plane waves in one dimension.

Following the procedure outlined for ion-acoustic waves, one finds that Langmuir waves of finite amplitude in a plasma with adiabatic electrons are also governed by the NS equation (2) with

$$P_{\alpha\beta} = \frac{1}{2\omega^3}[\delta_{\alpha\beta}(1+k^2) - k_\alpha k_\beta] \tag{15a}$$

and

$$Q = \frac{k^4}{3\omega}(16k^2 + 15), \tag{15b}$$

where ω and k satisfy the dispersion relation

$$\omega^2 = 1 + k^2.$$

In deriving the NS equation, ion dynamics has been neglected. In this case, ω is normalized to electron plasma frequency and k to $(4\pi N_0{}^2 e^2/\gamma p_0)^{1/2}$; γ is the ratio of specific heats and N_0 and p_0 are the equilibrium electron density and pressure, respectively. With the use of Eqs. (15a) and (15b), one can immediately show that the inequality (8a) cannot be satisfied for longitudinal modulation or for transverse modulation; thus the plane Langmuir waves are modulationally stable against long-wavelength perturbations as in the one-dimensional case. For a spherically symmetric case, however, Eq. (2) reduces to

$$i\frac{\partial\varphi}{\partial\tau} + \frac{1}{\omega}\left(\frac{1}{\mathcal{E}}\frac{\partial\varphi}{\partial\mathcal{E}} + \frac{1}{2\omega^2}\frac{\partial^2\varphi}{\partial\mathcal{E}^2}\right) = Q|\varphi|^2\varphi, \tag{16}$$

which in the linear case admits the solution

$$\varphi = b'\mathcal{E}^{\nu'} J_{\nu'}(k\mathcal{E}) \exp\left(-\frac{ik^2}{2\omega^3}\tau\right),$$

with

$$\nu' = -\tfrac{1}{2}(1+k^2) = -\tfrac{1}{2}(1+\omega^2 V_g{}^2).$$

In the limit $\vec{V}_g \to 0$, this represents a spherical wave. Once again, if I make the transformations, $\varphi = \rho^{\mu'}|Q|^{-1/2}\Psi$ and $\mathcal{E} = (2\omega^3)^{-1}\rho$, I can show that Eq. (16) has the following integrals of motion:

$$I_1' = \int_0^\infty d\rho\, \rho^2 |\Psi|^2$$

and

$$I_2' = \int_0^\infty d\rho\, \rho^2 \left[|\nabla\Psi|^2 + \frac{\mu'(\mu'-1)}{\rho^2}|\Psi|^2 + \tfrac{1}{2}\eta\rho^{2\mu'}|\Psi|^4\right].$$

Note that Q as defined by Eq. (15b) is positive definite and $\mu' = (1-\omega^2) = -k^2$. Moreover, I can show that $A' = \int_0^\infty d\rho\, \rho^4 |\Psi|^2$ satisfies the differential equation

$$\tfrac{1}{2} d^2A'/d\tau^2 = 12 I_2' - 8k^2(1+k^2)\int_0^\infty d\rho |\Psi|^2 - (3-2k^2)\eta \int_0^\infty d\rho\, \rho^{2(1-k^2)}|\Psi|^4.$$

Since $\eta = +1$ in this case, for $k < (\frac{3}{2})^{1/2}$, I obtain the relation

$$A' < 12 I_2' \tau^2 + C_1'\tau + C_2'.$$

As shown earlier, the collapse can occur only if $I_2 < 0$ which is possible only if $Q < 0$. Since in the case we are considering, $Q > 0$, we cannot have the collapse of Langmuir waves. This may seem to be in contradiction with the result obtained by Zakharov[3] but this is not so. Here I have considered only the noninertial nonlinearity because I have neglected the ion dynamics in the derivation of the NS equation whereas Zakharov in his model had taken only the nonlinearity that arises when the ion dynamics is important, i.e., in the strongly turbulent systems in which $\delta n_i/n \approx E^2/nKT \gtrsim k^2\lambda_D{}^2$, δn_i being the perturbation in the ion density and λ_D the electron Debye length. My analysis clearly shows that for a strongly turbulent system, one must consider both these nonlinearities which compete with each other; depending on which of the two nonlinearities is dominant, Q would be positive or negative which in turn will ascertain the possibility of occurrence of Langmuir collapse. For weakly turbulent systems, i.e., for $E^2/nKT \ll k^2\lambda_D^2$, noninertial nonlinearity, considered herein, dominates the nonlinearity arising because of ponderomotive force. It is worth pointing out that in Zakharov's[3] case, $Q<0$ which according to my analysis would correspond to a modulationally unstable case. Hence I assert once again that the collapse takes place only if the system is modulationally unstable and it is wrong to say that in three dimensions Langmuir waves always undergo collapse.

In conclusion, I would like to point out the two basic advantages of my method over the procedure followed by Zakharov.[3] Firstly Zakharov's method is not applicable to low-frequency waves, in particular ion-acoustic waves because of the averaging done on fast time scales. Secondly my method can consistently take care of both the nonlinearities mentioned above without neglecting the electronic nonlinearities and the second derivative in time as done by Zakharov.[3] The extension of present analysis, for strongly turbulent systems, including the ion dynamics, will be reported in a forthcoming paper.

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Ordered Helium Films on Highly Uniform Graphite—Finite-Size Effects, Critical Parameters, and the Three-State Potts Model*

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The heat capacity of the ordering transition of helium films is studied on a new graphite substrate whose microcrystallite allignment and lateral size are ten times that for Grafoil. Transition peaks are considerably sharper than those observed on Grafoil allowing determination of critical parameters $\alpha \cong 0.36$ and $A^+/A^- \cong 0.65$. The α is substantially larger than recent three-state Potts-model calculations in two dimensions (of which the ordering transition is the only example), raising the possibility that the ordered state belongs to a new universality class.

Considerable research has now been conducted with gases adsorbed on the substrate Grafoil, which is graphite exfoliated, and rolled from ground natural crystals.[1] Vapor pressure,[2] heat capacity,[3,4] neutron scattering,[5] NMR,[2,6] spreading pressure,[7] and other techniques have explored the monolayer and multilayer phases for a host of adsorbates on Grafoil. These studies relied on Grafoil's large surface area (20 m^2/g), partial crystallite orientation[5] (basal planes are $\lesssim 30°$ to sheet), and adequate thermal conductivity. However, the crystallites of this substrate appear to be quite small[5,7] ($\lesssim$ 200 Å) with a sizable fraction having a random orientation.[5] These features are known to have an appreciable effect on film behavior.[5,8] I report here the results of a heat capacity experiment performed on UCAR oriented graphite grade *ZYX* [1] which is carefully exfoliated from a stress-annealed pyrolytic graphite single crystal of monochronometer grade. Although the surface area is small and mass equilibrium times are long, the expanded *ZYX* is known to have an order of magnitude better crystallite orientation than Grafoil[9] ($\lesssim 3°$). This should be of considerable importance in enhancing signals for orientation-dependent experiments (NMR,[6] Mössbauer,[10] and neutron scattering[5]).

ZYX crystallites are surely far larger and more uniform than Grafoil.[9] To explore size effects I chose to investigate the ordering transi-