$O(\alpha)$ Corrections to the Decay Rate of Orthopositronium*

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The results of a new calculation of the $O(\alpha)$ corrections to the decay rate of orthopositronium are presented. The rate is $\Gamma=\Gamma^0[1-(\alpha/\pi)(10.348\pm 0.070)]=(7.0379\pm 0.0012)\times 10^6$ $\sec^{-1}$. This is substantially below all measured rates as well as previous theoretical estimates. We provide further justification for the computational techniques employed.

The decay rate of orthopositronium into three photons is the only decay rate of a purely quantum electrodynamic system that has been measured to an accuracy of better than 1%. The possibility suggested by recent experiments[1,3] of as much as a 2% discrepancy between existing theory and experiment necessitates a critical re-examination of the theory. In this Letter we present the results of a new calculation of all order-α correction to this decay rate. The complete problem was first considered by Stroscio and Holt.[3] Although we agree with the method of computation employed by these authors, our final result is considerably lower than their rate. We obtain

$$\Gamma_{o\text{-Ps}\to 3\gamma}=\Gamma^0[1-(\alpha/\pi)(10.348\pm 0.070)]$$
$$=(7.0379\pm 0.0012)\times 10^6\ \sec^{-1},$$

where Γ^0 is the lowest-order rate:

$$\Gamma^0=\frac{\alpha^6 m_e c^2}{\hbar}\frac{2(\pi^2-9)}{9\pi}=7.2112\times 10^6\ \sec^{-1}.$$

The measured rates are presented in Table I. The theoretical rate quoted above is inconsistent with all of the experimental rates. However the experimental situation is inconclusive at present because of the substantial difference between rates measured in SiO_2 powder[1] and in *vacua*[2] and those measured in gases.[4,5] Should the difference between theory and experiment persist it will be necessary to compute corrections of orders $\alpha^2\ln\alpha$ and α^2.

The three-photon decay amplitude for positronium is

$$T(k_i)=\int d^4p\,(2\pi)^{-4}\psi_{\rm BS}(K,p)i\mathfrak{M}(K,p,k_i), \tag{1}$$

where $\psi_{\rm BS}$ is the Bethe-Salpeter wave function and $\mathfrak{M}$ is the two-particle irreducible electron-positron decay kernel. Adopting the usual perturbative treatment, we replace $\psi_{\rm BS}$ by $\psi_{\rm BS}{}^{\rm C}$, the solution of the Bethe-Salpeter equation with an instantaneous Coulomb kernel[6]:

$$(\tfrac{1}{2}K\!\!\!/+p\!\!\!/-m_e)(p\!\!\!/-\tfrac{1}{2}K\!\!\!/-m_e)\psi_{\rm BS}{}^{\rm C}(K,p)$$
$$=-ie^2\int\frac{d^4q}{(2\pi)^4}\frac{\gamma^0\gamma^0}{|\vec{p}-\vec{q}|^2}\psi_{\rm BS}{}^{\rm C}(K,q). \tag{2}$$

For $|\vec{p}|\ll m_e$, $\psi_{\rm BS}{}^{\rm C}$ becomes (in the atom's rest frame where $K^0\simeq 2m_e-\alpha^2 m_e/4$)[7]

$$\psi_{\rm BS}{}^{\rm C}(K,p)$$
$$\simeq\frac{2\pi}{i}\delta\left(p^0-\frac{\vec{p}^2}{2m_e}\right)(2K^0)^{1/2}u(\vec{p})\bar{v}(-\vec{p})\psi_{\rm NR}(\vec{p}),$$

where $\psi_{\rm NR}$ is the nonrelativistic Schrödinger wave function:

$$\psi_{\rm NR}(\vec{p})=[8\pi\gamma/(\vec{p}^2+\gamma^2)^2]\psi_0,$$
$$\psi_0=\psi_{\rm NR}(\vec{x}=0)=(\gamma^3/\pi)^{1/2},$$

and $\gamma=\alpha m_e/2$.

The lowest- and first-order terms in the orthopositronium decay rate result from the kernels in Fig. 1. It is important that the decay kernel contains all interactions not already included in the wave function. Thus graph (g) in which a transverse photon is exchanged by the electron and positron must be considered. The instantaneous Coulomb interaction is part of the wave function.

The only contributions to order $\alpha\Gamma^0$ from graphs (b) through (g) come from the region of small relative momentum $[p\sim O(\gamma)]$ in Eq. (1). Also, the effects of binding in the decay kernel are negligible here. Thus the decay amplitude may be expressed in terms of the nonrelativistic wave function and the real part of the electron-positron annihilation amplitude (which includes the spin

TABLE I. Experimental determinations of the decay rate of orthopositronium into three photons.

Ref.	Rate (10^6 $\sec^{-1}$)	Deviation from theory
1	7.104 ± 0.006	$(3.9\pm 0.4)(\alpha/\pi)\Gamma^0$
2	$7.09\ \ \pm 0.02$	$(3\pm 1)(\alpha/\pi)\Gamma^0$
4	7.262 ± 0.015	$(13.4\pm 0.9)(\alpha/\pi)\Gamma^0$
5	7.275 ± 0.015	$(14.1\pm 0.9)(\alpha/\pi)\Gamma^0$

factors) evaluated on mass shell:

$$T_{\text{b-g}}(k_i) = (4m_e)^{1/2}\int d^3p\,(2\pi)^{-3}\psi_{\text{NR}}(\vec{p})\,\text{Re}[\mathfrak{M}_{\text{b-g}}{}^{\text{MS}}(\vec{p},k_i)] \simeq (4m_e)^{1/2}\psi_0\,\text{Re}[\mathfrak{M}_{\text{b-g}}{}^{\text{MS}}(0,k_i)],$$

where K^0 has been replaced by $2m_e$. The imaginary part of the amplitude vanishes below threshold and therefore should be omitted. Also, $\mathfrak{M}_{\text{b-g}}$ is well behaved for $p \sim O(\gamma) \ll m_e$ and may be replaced by its value at threshold ($\vec{p}=0$) to the relevant order in α.

The lowest-order contribution to the decay rate comes from graph (a) which gives Γ^0 in the small-p regime. Contrary to the statements made in Ref. 3, relativistic corrections from the wave function and from the propagators in this graph give corrections of order α to the amplitude. Generally these corrections take the form

$$\delta T_{\text{a}} \propto (I^0)^{1/2}\int d^3p[\gamma/(\vec{p}^2+\gamma^2)^2][f(\vec{p})-f(0)], \tag{3}$$

where $f(p)$ has no explicit dependence on α. As argued in Ref. 3, $f(p)$ may be expanded in a power series in $\vec{p}/m_e$ for $p \sim O(\gamma) \ll m_e$. Since terms linear in $\vec{p}$ integrate to zero, the leading contribution to $f(\vec{p})-f(0)$ is proportional to $\vec{p}^2/m_e^2 \sim O(\alpha^2)$ *for p nonrelativistic.* Thus for $p \sim O(\gamma)$ there is only an $O(\alpha^2)$ correction to the amplitude. However the operator $\vec{p}^2/m_e^2$ leads to a linear divergence as $p \to \infty$ when introduced in δT_{a}. This indicates that the dominant contribution to δT_{a} comes from the relativistic regime $[p \sim O(m_e)]$ where a Taylor expansion is inappropriate. For $p \sim O(m_e)$, the integrand in Eq. (1) is of order α/m_e^3 and d^3p is of order m_e^3. Therefore δT_{a} is of order $\alpha(\Gamma^0)^{1/2}$, resulting in an $O(\alpha)$ correction to the rate.

These $O(\alpha)$ corrections from graph (a) are most easily computed in conjunction with those from graph (g).[8] Simple power counting arguments, similar to those used above, together with the Bethe-Salpeter equation [Eq. (2)] assure that evaluating kernels (a) and (g) with the Bethe-Salpeter wave function is completely equivalent in our order of approximation to evaluating graph (g′) on mass shell with the nonrelativistic wave function (Fig. 1). Graph (g′) is identical to (g) but with the transverse photon propagator replaced by the complete photon propagator. Binding corrections in $\mathfrak{M}_{\text{g}'}$ are $O(\alpha^2)$ and may be safely ignored.

Thus the entire decay amplitude including radiative corrections of $O(\alpha)$ can be expressed in terms of electron-positron annihilation amplitudes evaluated on mass shell and the nonrelativistic Schrödinger wave function:

$$T_{o\text{-Ps}\to 3\gamma}(k_i) = (4m_e)^{1/2}\psi_0\,\text{Re}[\mathfrak{M}_{\text{b-f}}{}^{\text{MS}}(\vec{p}=0,k_i)] + (4m_e)^{1/2}\int d^3p\,(2\pi)^{-3}\psi_{\text{NR}}(\vec{p})\,\text{Re}[\mathfrak{M}_{\text{g}'}{}^{\text{MS}}(\vec{p},k_i)].$$

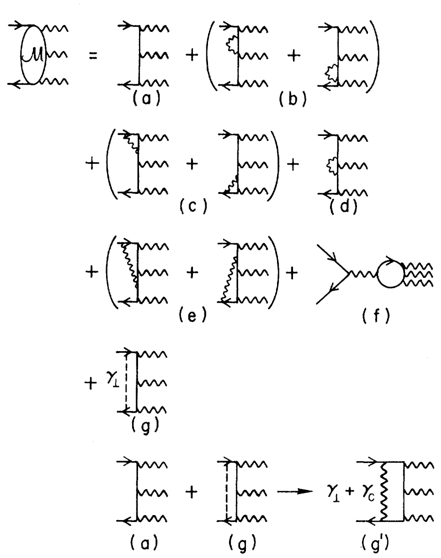

FIG. 1. The orthopositronium decay kernel contributing to $O(\alpha\Gamma^0)$. Graphs (a) and (g) may be replaced by (g′).

Clearly this method of computation is gauge independent to this order and we are free to evaluate $\mathfrak{M}$ in the Feynman gauge.

The $O(\alpha)$ corrections to the rate from graphs (b), (c), and (d) are

$$\Gamma_{\text{b}} = [4.791 \pm 0.003 + 4\ln(\lambda/m_e)](\alpha/\pi)\Gamma^0,$$

$$\Gamma_{\text{c,d}} = [-2.868 \pm 0.003 - 6\ln(\lambda/m_e)](\alpha/\pi)\Gamma^0.$$

These agree with the results in Ref. 3, exhibited here in Table II. The diagrams were renormalized on mass shell in the usual fashion.

Kernels (e), (f), and (g′) were computed independently by each of the authors. Two methods were employed to perform the loop integrations. In one, standard Feynman-parameter techniques were used, while in the other, integrations over the loop momenta (k) were performed directly. For the latter technique, the k^0 contour was closed at infinity and the residues of the propagator poles computed. All results agreed. The con-

TABLE II. Theoretical determinations of the $O(\alpha)$ corrections to the decay rate of orthopositronium [in units of $(\alpha/\pi)\Gamma^0$]. Infrared infinite terms have been omitted.

	This paper	Ref. 3
Γ_b	4.791 ± 0.003	4.785 ± 0.010
$\Gamma_{c,d}$	-2.868 ± 0.003	-2.8716 ± 0.0036
Γ_e	-3.562 ± 0.004	-3.355 ± 0.003
Γ_f[a]	-0.809 ± 0.004	-0.5 ± 0.2
$\Gamma_{g'}$	-7.90 ± 0.07	3.8 ± 0.4
Total	-10.348 ± 0.070	1.86 ± 0.45

[a] $\Gamma_f = -0.741 \pm 0.017$ is quoted in Ref. 9.

tributions from graphs (e) and (f) are

$$\Gamma_e = (-3.562 \pm 0.004)(\alpha/\pi)\Gamma^0,$$

$$\Gamma_f = (-0.809 \pm 0.004)(\alpha/\pi)\Gamma^0.$$

These numbers are in slight disagreement with Ref. 3. The second rate also disagrees slightly with that quoted in Ref. 9.

In computing graph (f) it is necessary to regulate the loop integration if gauge invariance is to be satisfied. Pauli-Villars regulation was used as well as the technique described by Aldins *et al.*,[10] where the vacuum polarization tensor is replaced by

$$\Pi_{\mu\nu\rho\sigma}(k_1k_2k_3k_4) \to -k_1^{\alpha}\frac{\partial\Pi_{\alpha\nu\rho\sigma}(k_1k_2k_3k_4)}{\partial k_1^{\mu}}.$$

As a check, the rate due to a heavy fermion (M) loop was computed and found to agree with the analytic result[9]:

$$\Gamma_M = -\frac{11}{135}\frac{29-3\pi^2}{9-\pi^2}\left(\frac{m_e}{M}\right)^4\frac{\alpha}{\pi}\Gamma^0.$$

Graph (g′) has both a logarithmic singularity and a $1/|\vec{p}|$ singularity at threshold. These may be removed from $\mathfrak{M}_{g'}$ by subtracting the quantity

$$\mathfrak{M}_{g'}^{\mathrm{IR}} = ie^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-\lambda^2+i\epsilon}\,\frac{1}{k^2+2p_1\cdot k+i\epsilon}\,\frac{1}{k^2-2p_2\cdot k+i\epsilon}\,4m_e^2\mathfrak{M}_a^{\mathrm{MS}}(\vec{p}=0,k_i)$$
$$=\frac{\alpha}{2\pi}\left[2\ln\left(\frac{\lambda}{m_e}\right)+\frac{\pi^2 m_e}{2p}-2\right]\mathfrak{M}_a^{\mathrm{MS}}(\vec{p}=0,k_i),$$

where $p_1=\frac{1}{2}K+p$ and $p_2=\frac{1}{2}K-p$. For reasons discussed above, the imaginary Coulomb phase is omitted. The amplitude $\mathfrak{M}_{g'}^{\mathrm{IR}}$ contributes

$$\Gamma_g^{\mathrm{IR}} = \Gamma^0 + [2\ln(\lambda/m_e)-2](\alpha/\pi)\Gamma^0$$

to the rate. The subtracted amplitude, though finite, is difficult to evaluate numerically because of the branch point at threshold due to electron-positron intermediate states. To overcome this difficulty in the Feynman-parameter treatment, the amplitude was evaluated at several points below threshold and an extrapolation to threshold made. When the loop momenta were integrated directly, the problem was avoided by cutting off the integration in the small-k region. The final result was computed by extrapolating to zero cutoff. Again all results agreed, the subtracted rate being

$$\Gamma_{g'}^{s} = (-5.90 \pm 0.07)(\alpha/\pi)\Gamma^0.$$

Thus the total contribution from graph g′ is

$$\Gamma_{g'} = \Gamma^0 + [2\ln(\lambda/m_e) - 7.90 \pm 0.07](\alpha/\pi)\Gamma^0$$

The numerical constant is in disagreement with that of Ref. 3.[11]

All gamma matrix manipulations were performed by Hearn's program REDUCE.[12] The integrals were evaluated numerically using Monte Carlo integration programs by Sheppey[13] and by Lepage.[14] The uncertainties quoted above for the theoretical rate are the standard deviations computed by these programs.

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[6]Actually this procedure is poorly suited to positronium as the annihilation potential [$\sim\delta(x)$] leads to infinities when treated perturbatively [$\psi_{\mathrm{BS}}^{C}(x=0)=\infty$]. Annihilation should be incorporated into the unperturbed potential in some way. This modifies our treatment only at the $O(\alpha^2\Gamma^0)$ level and so is irrelevant here. A more thorough examination of this problem is under way in collaboration with S. J. Brodsky.

[7]The conventions of J. D. Bjorken and S. D. Drell,

Relativistic Quantum Mechanics (McGraw-Hill, New York, 1964), have been adopted in this paper.

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[11]Note that changing the sign of $\Gamma_g{}^{,S}$ leads to agreement with Ref. 3 for this graph. To check for sign errors in our computer programs we used them to compute Γ_g without infrared subtractions. The results diverged linearly to $+\infty$ as would be expected from the $1/p$ term in $\Gamma_g{}^{,IR}$.

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Experimental Measurement of Electron Heat Diffusivity in a Tokamak*

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Electron temperature perturbations produced by internal disruptions in the center of the Oak Ridge Tokamak (ORMAK) are followed with a multichord soft-x-ray detector array. The space-time evolution is found to be diffusive in character, but the conduction coefficient determined from a heat-pulse-propagation model is larger by a factor of 2.5–15 than that implied by the measured gross energy-containment time.

A useful model for understanding the energy transport governing the behavior of tokamak discharges is a three-region plasma model. The central-core region ($r < a_D$, the disruption radius) suffers internal disruptions[1] repeatedly as the safety factor q drops below unity. Outside this core region there is typically a large "middle" region (confinement zone) where tearing modes, plasma turbulence and/or unknown processes are responsible for "anomalous" heat transport, which primarily determines the energy containment of the device. Finally, there is a "plasma-edge" region ($r > a_0$) dominated by atomic physics effects such as radiation, impurity refluxing, charge exchange, etc.

The internal disruptions inside a_D manifest themselves as sudden drops in the soft-x-ray signal level, followed by slower recoveries, giving the characteristic sawtooth pattern evident in Fig. 1. The standard interpretation[1] of the sudden drop is that the electron temperature is decreasing as heat is rapidly lost from the central region. This process, which we will not discuss in detail, results in a pulse of heat into the volume just outside the disruption radius, and predictably, as seen in Fig. 1, the x-ray signals outside a_D show a pulselike increase at the time of the sudden decrease inside. By following the propagation of these perturbations through the critical middle region, we can, directly and for the first time, examine the fundamental electron-heat-transport process in tokamaks.

The soft-x-ray system on ORMAK consists of nine silicon diffused-junction diode detectors that view different fixed chords through the plasma.[2] The x-ray signal results from plasma bremsstrahlung and recombination processes, both of which are strongly dependent on tempera-

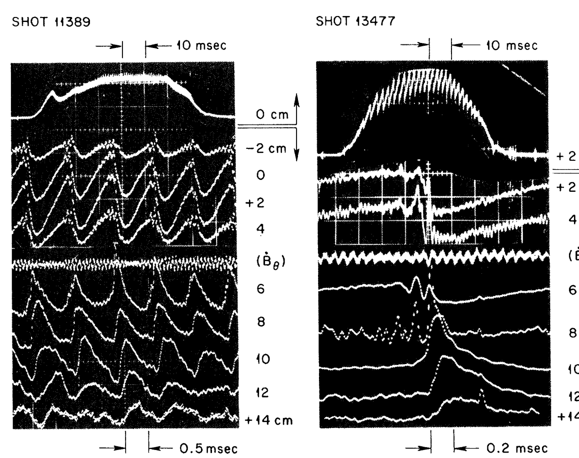

FIG. 1. Composite oscillograms of soft-x-ray signals for two discharges. For both cases, the top trace gives the signal from one detector over the full time of the discharge; the rest of the signals are on an expanded timescale starting at 45 msec, which falls in the middle of the full-time trace. Amplification factors are different for purposes of display. The temporal variation in the signal (sharp fall inside, sharp rise outside) shows that a_D is $\simeq 5$ cm for shot 11 389 and $\simeq 8$ cm for shot 13477. The signals labeled $\dot{B}_\theta$ are poloidal magnetic field fluctuations from pickup loops.