Radiative Capture and Galilean Invariance

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Treating the radiation field as a first-order perturbation, Briggs and Dettmann recently studied radiative capture. The cross section, calculated approximately, depended on the frame of reference chosen. By use of natural nonrelativistic limit, it is shown here that if the approximate initial and final unperturbed wave functions are orthogonal—as the exact ones are—then Galilean invariance holds; otherwise, Galilean invariance is, in general, invalidated because of spurious radiation from the center of mass of the system.

We will here be concerned with a cross section $\Delta\sigma_{\rm rad}$ (obtained, for a fixed initial state, by integrating over some set of final state) associated with radiative capture or its inverse [an example being the three-body case $A+(B+C)\rightleftharpoons(A+B)+C+\gamma$], or radiative recombination or its inverse [an example being the two-body case $A+B\rightleftharpoons(A+B)+\gamma$]. Like the nonradiative cross section, $\Delta\sigma_{\rm rad}$ will be Lorentz-invariant in a fully relativistic treatment. However, the particles (that is, those with nonzero rest mass) can often be accurately described by the nonrelativistic Galilean-invariant Schrödinger equation, and so the question naturally arises as to whether, in the nonrelativistic limit, the radiative cross section is Galilean-invariant. The difficulty, of course, lies in the presence of the photon, which is defined by the Lorentz-invariant (not Galilean-invariant) Maxwell equations. The question is really whether there is a sense in which the photon can sensibly be treated nonrelativistically. In the problems under consideration we can think of the energy E_γ of the photon in any given frame as independent of c; E_γ is the difference between the initial and final energies of the particles, energies which have a well-defined nonrelativistic limit. The wave number $k=E_\gamma/\hbar c$ of the photon therefore vanishes as $c\sim\infty$, and in the interaction of the photon with the particles we can set the photon momentum $\hbar\vec{k}$ equal to zero, that is, we make the dipole approximation. (Actually, we do not set $\vec{k}$ equal to zero until after the center-of-mass coordinate has been integrated over, since $\vec{k}$ appears multiplied by this coordinate, which is unbounded.) Note that since the incident systems [A and B, or A and $(B+C)$, for example] have all possible relative orbital angular momenta, dipole radiation is always possible. It is consistent with the dipole approximation to neglect, under a transformation of frames, the Doppler shift in the frequency and the change in the polarization vector of the photon. We might expect that with these approximations the cross section is Galilean-invariant.

The interaction $H_{\rm int}$ of the particles with the radiation field will throughout be treated as a first-order perturbation; $H_{\rm int}$ appears as an operator in a matrix element, but has no influence on the wave functions. The wave functions are taken to be the exact solutions of the (nonrelativistic) Schrödinger equation, and $H_{\rm int}$ is taken to be that appropriate to the interaction of radiation with nonrelativistic particles. We will refer to the above approximation as the nonrelativistic first-order perturbation (NF) approximation, and will denote the radiative cross section obtained in that approximation by $\Delta\sigma_{\rm rad}{}^{\rm NF}$. The NF cross section obtained by using approximate wave functions will be denoted by $\Delta\tilde{\sigma}_{\rm rad}{}^{\rm NF}$.

Recently, Briggs and Dettmann[1] investigated the form of $\Delta\tilde{\sigma}_{\rm rad}{}^{\rm NF}$ for the radiative capture of an electron by a bare nucleus incident at an asymptotically high velocity on a hydrogenlike ion or atom. They found that the NF approximation, with the wave functions determined in the first Born approximation, gave the correct asymptotic form for $\Delta\sigma_{\rm rad}{}^{\rm NF}$ when applied in the frame of the projectile but the incorrect asymptotic form when applied in the frame of the target. The purpose of this Comment is twofold: First, to give the reason (which is apparently not well known) for the frame dependence of $\Delta\sigma_{\rm rad}{}^{\rm NF}$ in the circumstance just indicated; and, second, to indicate the conditions under which $\Delta\sigma_{\rm rad}{}^{\rm NF}$ and $\Delta\tilde{\sigma}_{\rm rad}{}^{\rm NF}$ *are*

Galilean-invariant.

We begin by discussing the radiative recombination process $m_1+m_2\to(m_1+m_2)+\gamma$. We denote the particles by their masses, and for simplicity assume them to be spinless. The charges are taken to be q_1 and q_2, with at least one of the charges nonzero since it must be possible to radiate a photon; there may be a non-Coulombic interaction between the two particles. Let $\vec{x}_1$ and $\vec{x}_2$ be the particle coordinates relative to some arbitrary origin 0; and let $\vec{x}\equiv\vec{x}_2-\vec{x}_1$ and $\vec{X}$, respectively, be the relative and center-of-mass (c.m.) coordinates. Let $\vec{p}_1$ and $\vec{p}_2$ be the momentum operators of the particles; and let $\vec{p}$ and $\vec{P}$ be the relative and c.m. momentum operators. Let $\vec{\lambda}$ denote the polarization vector of the field. By omitting terms associated with absorption, which do not contribute to the process under consideration, $H_{\rm int}$ becomes, in an inertial frame whose origin is 0, the operator

$$H_{\rm int}=a\omega^{1/2}\vec{\lambda}\cdot\sum_i q_i(\vec{p}_i/m_i)\exp(-i\vec{k}\cdot\vec{x}_i)=a\omega^{-1/2}\exp(-i\vec{k}\cdot\vec{X})\sum_i\vec{\lambda}\cdot\vec{B}_i, \tag{1a}$$

where i is to be summed over 1 and 2 and where

$$\vec{B}_1=q_1(\vec{P}/M-\vec{p}/m_1)\exp[i(m_2/M)\vec{k}\cdot\vec{x}], \tag{1b}$$

$$\vec{B}_2=q_2(\vec{P}/M+\vec{p}/m_2)\exp[-i(m_1/M)\vec{k}\cdot\vec{x}]. \tag{1c}$$

Here, $M\equiv m_1+m_2$ is the total mass of the system, and a is a constant. The primary point of interest is that $H_{\rm int}$ depends not only on the relative momentum $\vec{p}$ but also on the c.m. momentum $\vec{P}$, that is, that $H_{\rm int}$ is *not* Galilean-invariant. The exact initial and final wave functions of the system are

$$\exp(i\vec{K}_i\cdot\vec{X})\Phi_i(\vec{x})\ \text{and}\ \exp(i\vec{K}_f\cdot\vec{X})\Phi_f(\vec{x}), \tag{2}$$

where $\hbar\vec{K}_i$ and $\hbar\vec{K}_f$ are, respectively, the initial and final c.m. momenta, and where $\Phi_i(\vec{x})$ and $\Phi_f(\vec{x})$ are, respectively, the exact initial continuum and final bound-state wave functions that describe the internal motion. The matrix element of $H_{\rm int}$, taken between the wave functions of Eq. (2), is

$$\langle H_{\rm int}\rangle_{fi}=a\omega^{-1/2}(2\pi)^3\delta(\vec{K}_i-\vec{k}-\vec{K}_f)\mathfrak{M},$$

where, with $\vec{u}=\hbar\vec{K}_i/M$ the initial velocity of the center-of-mass relative to 0,

$$\mathfrak{M}=\vec{\lambda}\cdot\vec{u}\langle\Phi_f|\{q_1\exp[i(m_2/M)\vec{k}\cdot\vec{x}]+q_2\exp[-i(m_1/M)\vec{k}\cdot\vec{x}]\}|\Phi_i\rangle$$
$$+\vec{\lambda}\cdot\langle\Phi_f|\{(q_2/m_2)\exp[-i(m_1/M)\vec{k}\cdot\vec{x}]-(q_1/m_1)\exp[i(m_2/M)\vec{k}\cdot\vec{x}]\}\vec{p}|\Phi_i\rangle. \tag{3}$$

In arriving at this form, we have used $\vec{\lambda}\cdot\vec{k}=0$. Note that we can replace $\vec{k}$ by the Galilean-invariant vector $\vec{K}_i-\vec{K}_f$. At this point we make the dipole approximation, that is, we set $\vec{k}$ equal to zero. With $Q\equiv q_1+q_2$ denoting the total charge of the system, we arrive at

$$\langle H_{\rm int}\rangle_{fi}=a\omega^{-1/2}(2\pi)^3\delta(\vec{K}_f-\vec{K}_i)\vec{\lambda}\cdot[Q\vec{u}\langle\Phi_f|\Phi_i\rangle+(q_2/m_2-q_1/m_1)\langle\Phi_f|\vec{p}|\Phi_i\rangle]. \tag{4}$$

At first sight, $\langle H_{\rm int}\rangle_{fi}$, and hence $\Delta\sigma_{\rm rad}^{\rm NF}$, appear to depend on $\vec{u}$. The term that contains $\vec{u}$ arises from the center-of-mass current operator. However, noting that in the dipole approximation the center of mass does not recoil, the current generated by the center of mass is constant in time and therefore cannot radiate. The $\vec{u}$ term should therefore be zero. This term is, in fact, zero, since Φ_i and Φ_f are eigenfunctions of the same Hamiltonian with different energy eigenvalues and are therefore orthogonal.[2] The relevant entity in the evaluation of the radiative cross section is therefore $\omega^{-1/2}\vec{\lambda}\cdot\langle\Phi_f|\vec{p}|\Phi_i\rangle$. Since $\vec{p}$ is a relative momentum, and since Φ_i and Φ_f depend only on the relative coordinate $\vec{x}$, the quantity $\langle\Phi_f|\vec{p}|\Phi_i\rangle$ is invariant under a Galilean transformation. Therefore, if we make the dipole approximation, and neglect the Doppler shift in ω and the change in $\vec{\lambda}$, $\Delta\sigma_{\rm rad}^{\rm NF}$ is independent of $\vec{u}$, and hence of the inertial frame used, so that it is Galilean-invariant. Suppose, however, that the exact wave functions Φ_i and Φ_f are replaced by approximations $\tilde{\Phi}_i$ and $\tilde{\Phi}_f$. If $\langle\tilde{\Phi}_f|\tilde{\Phi}_i\rangle$ is not zero, then, in general, the $\vec{u}$ term is no longer zero, the center of mass radiates spuriously and $\Delta\tilde{\sigma}_{\rm rad}^{\rm NF}$ is not Galilean-invariant. An exception arises when $Q=0$, for then the system is electrically neutral and the center of mass cannot generate any current so that the possibility of spurious radiation does not arise. However, if $\langle\tilde{\Phi}_f|\tilde{\Phi}_i\rangle\neq0$ and $Q\neq0$, the coefficient of $\vec{u}$ is nonzero and, in order to eliminate spurious radiation and avoid obtaining a completely erroneous result, one must either neglect the term in $\vec{u}$ or work in the frame in which there is no center-of-mass current, that is, in the center-of-mass frame, where $\vec{u}=0$.

Of the various approximations, the Born approximation to be denoted by the subscript B, is a particularly interesting one. We use an arbitrary approximation $\tilde{\Phi}_f$ for the bound-state wave function Φ_f, but we replace the continuum wave function Φ_i by the plane wave $\tilde{\Phi}_{i\,\mathrm{B}}=\exp(i\mu\vec{v}\cdot\vec{x}/\hbar)$, where μ is the reduced mass and $\vec{v}$ is the relative velocity. The expression in square brackets in Eq. (4) then becomes $\vec{D}_{\mathrm{B}}\langle\tilde{\Phi}_f|\tilde{\Phi}_{i\,\mathrm{B}}\rangle$, where $\vec{D}_{\mathrm{B}}\equiv Q\vec{u}+[(q_2m_1-q_1m_2)/M]\vec{v}$. In this approximation, $\langle H_{\mathrm{int}}\rangle_{fi\,\mathrm{B}}$, and hence $\Delta\tilde{\sigma}_{\mathrm{rad\,B}}{}^{\mathrm{NF}}$, vanish identically in the frame in which $\vec{D}_{\mathrm{B}}=0$—a strong indication that some care is required in the choice of frame. Now the radiative capture process $M_B+(m+M_A)\to(M_B+m)+M_A+\gamma$, where an electron of mass m and charge e is initially bound to a nucleus of mass M_A and charge $-Z_Ae$ and is captured by a nucleus of mass M_B and charge Z_Be, is essentially the radiative recombination process $M_B+m\to(M_B+m)+\gamma$ at asymptotically high velocities; the binding energy of m and M_A and the distribution of their initial relative momentum can be neglected and therefore M_A plays no significant role.[3] Setting $m_1=M_B$, $q_1=-Z_Be$, $m_2=m$, and $q_2=e$, and neglecting terms of order m/M_B, $\vec{D}_{\mathrm{B}}$ becomes $\vec{D}_{\mathrm{B}}\approx e[(1-Z_B)\vec{u}+\vec{v}]$. Now Briggs and Dettman use the Born approximation for the radiative capture process under consideration, but since they omit the currents generated by the nuclei no terms in Z_B appear.[4] In this further approximation, and to the extent that the radiative-recombination and radiative-capture processes are equivalent, the capture cross section vanishes for $\vec{u}=-\vec{v}$, that is, in the lab frame, in agreement with their result.[1]

For the first time, we now drop the assumption that the photon carries no momentum, but we continue to make the NF approximation. Relativistic corrections are of order $1/c^2$ so it would be inconsistent to include corrections beyond first order in $1/c$ due to the photon momentum. If photon momentum corrections are included only to first order in $1/c$, the total radiative-recombination cross section $\sigma_{\mathrm{rad}}{}^{\mathrm{NF}}$ remains Galilean-invariant, and the approximation $\tilde{\sigma}_{\mathrm{rad}}{}^{\mathrm{NF}}$ to $\sigma_{\mathrm{rad}}{}^{\mathrm{NF}}$ remains Galilean-invariant if $\langle\tilde{\Phi}_f|\tilde{\Phi}_i\rangle=0$ and/or $Q=0$, since correction terms of order $1/c$ vanish upon integration over all directions of emission. (Note that when we ignored corrections of order $1/c$, $\Delta\sigma_{\mathrm{rad}}{}^{\mathrm{NF}}$ and $\Delta\tilde{\sigma}_{\mathrm{rad}}{}^{\mathrm{NF}}$ were Galilean-invariant for any fixed $\vec{\lambda}$ and for integration over any range of the angle θ at which the photon is emitted; we did not require summation over $\vec{\lambda}$ and integration over all θ to achieve Galilean invariance.)

It is a simple matter to extend the above argument concerning Galilean invariance to the radiative-capture process $M_B+(m+M_A)\to(M_B+m)+M_A+\gamma$ when the initial binding of m and M_A does play a role. No spurious radiation is produced, and the cross section is Galilean-invariant, provided that the initial and final internal wave functions (exact or approximate) of $(m+M_A+M_B)$ are orthogonal. Note that approximate internal wave functions that are chosen to be exact solutions of the coupled-state (or close-coupling) equations are orthogonal, since the solutions of these equations are eigenfunctions, with different energy eigenvalues, of the same given (model) Hermitian Hamiltonian.

It may be of interest to comment on the fact that the nonrelativistic limit of H_{int}, in the present context, is the dipole approximation. The point, of course, is that since the electromagnetic field is a vector field, and since H_{int} is a scalar, the particle-dependent factors are vectors ($\vec{p}_1$ and $\vec{p}_2$, or $\vec{p}$ and $\vec{P}$), allowing a change of one unit of angular momentum when the exponentials are set equal to unity—this is just the intrinsic spin of the photon. To get a change of more than one unit of angular momentum, the exponentials cannot be set equal to unity since the photon must have orbital angular momentum and therefore linear momentum.[5] This point is well known but it is not usually stated as the "nonrelativistic limit of the photon." (For quadrupole or higher-multipole processes, we do not let $c\to\infty$ but simply retain the leading term in $1/c$; $\Delta\sigma_{\mathrm{rad}}{}^{\mathrm{NF}}$ will then again be Galilean-invariant.)

Note that the entire discussion above can be extended to the impact-parameter approximation in which the heavy nuclei are treated as classical particles. The condition for Galilean invariance becomes

$$I=\int_{-\infty}^{\infty}\exp(i\omega t)\langle\Phi_f(t)|\Phi_i(t)\rangle dt=0;$$

the spatial integration is now only over the coordinates of those particles which are being treated quantum mechanically. (When all of the particles are treated quantum mechanically, as in the discussion above, the time dependence of the wave functions is oscillatory and the time integration can be performed immediately to produce an energy-conserving δ function. With the time dependence factored out of the wave functions, the condition for I to vanish is that $\langle\Phi_f|\Phi_i\rangle$ vanish, which is the same condition as above.) Energy is not conserved within the impact-parameter approximation, and $\Phi_f(t)$ and $\Phi_i(t)$ are not orthogonal. However, since ω is nonzero, I will vanish if $\langle\Phi_f(t)|\Phi_i(t)\rangle$ is independent of the time t, which it is since $\Phi_i(t)$ and $\Phi_f(t)$ satisfy a time-dependent Schrödinger equation whose Hamiltonian is Hermitian. This remains true if $\Phi_i(t)$ and $\Phi_f(t)$ are replaced by approximate wave functions that satisfy the time-dependent coupled-state equations, since this amounts to replacing the true Hamiltonian by a model Hamiltonian which is also Hermitian. The fact that Galilean invariance will be preserved if $\langle\tilde{\Phi}_f(t)|\tilde{\Phi}_i(t)\rangle$ is constant in time has been noted previously by Briggs and Dettmann.[6]

All of the remarks above should be applicable to bremstrahlung processes. Furthermore, for radiative processes which have a classical meaning, the cross section, calculated classically, would be expected to have a Galilean-invariant nonrelativistic limit.

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[1]J. S. Briggs and K. Dettmann, Phys. Rev. Lett. 33, 1123 (1974).

[2]In the formal derivation of cross sections there appears a δ function which represents conservation of energy. In analyzing the possiblity of radiation by a free particle, the energy-conserving δ function appears multiplied by a momentum-conserving δ function. The impossibility of both arguments simultaneously vanishing precludes the possiblity of radiation by a free particle. In the present case in which radiation is in fact generated by two interacting particles, radiation is obviously not precluded by considerations of energy and momentum conservation by virtue of the binding energy of the final state; radiation related to the motion of the center of mass is precluded, but by the vanishing of the internal coordinate matrix element.

[3]See, for example, G. Raisbeck and F. Yiou, Phys. Rev. A 4, 1858 (1971); H. W. Schnopper, H. D. Betz, J. P. Delvaille, K. Kalata, and A. R. Sohval, Phys. Rev. Lett. 29, 898 (1972). Some physical insights into the various capture processes (both radiative and nonradiative) at asympototically high velocities, are provided in articles to be published by the authors.

[4]Nuclear currents can be neglected in the impact-parameter approximation, since the nuclei are then assumed to move with constant velocity and will therefore not radiate.

[5]The discussion of this paragraph should not be taken too literally since the photon's orbital angular momentum cannot be separated from its spin angular momentum owing to the impossibility of bringing the photon to rest to measure its spin.

[6]J. S. Briggs, private communication. See also K. Dettmann and J. S. Briggs, in *Proceedings of the Ninth International Conference on the Physics of Electronic and Atomic Collisions, Seattle, Washington, 1975,* edited by J. S. Risley and R. Geballe (Univ. of Washington Press, Seattle, 1975).

Comments on "Host Nuclear Resonance in a Spin-Glass: *Cu*Mn"

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New data on a higher concentration (~1%) of Mn in Cu than that studied by MacLaughlin and Alloul exhibit fundamentally different behavior for $T \ll T_c$, where $T_c \sim 10$ K is the "spin-glass" transition temperature. In particular, we find evidence for a "frozen" configuration of Mn spins at $T = 1.6$ and 4.2 K, in better accord with the muon-precession studies of Murnick *et al*.

Recently MacLaughlin and Alloul[1] (MA) published a study of ^{63}Cu NMR linewidths and relaxation times in dilute (0.1–0.4%) *Cu*Mn alloys at temperatures in the vicinity of the spin-glass ordering temperature T_c. Curiously anomalous results were reported by MA. The inhomogeneous linewidth showed no detectable change at T_c, in contrast with the dramatic increase exhibited in the muon-precession experiments of Murnick, Fiory, and Kossler.[2] Furthermore, even at the lowest temperatures studied (~ 1.5 K), the observed ^{63}Cu linewidths scaled with applied field H_0 to within experimental error. Thus, there was no evidence for a "frozen-in" state of Mn