[7]The choice of plotting the half-maximum is arbitrary. Plots similar to those shown in Fig. 2 are obtained for the change in positions of quarter- (three-quarter-) maximum except that I_D and n_D shift to lower (higher) values.

[8]The calculated plasma curve is shifted along the energy axis to obtain the best fit.

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[13]We have no experimental evidence to support this assumption. However, we find that the exciton luminescence intensity varies linearly with I. If the volume occupied by the excitations at 70-nsec delay varies sufficiently slowly with I and T, deviations from linear behavior may introduce effects within the experimental error bars. Note also that I_D and I_T are at least 30 times smaller than the laser power used in Ref. 5. Hence the initial density and expansion rate are considerably smaller. At 70-nsec delay we find that complete thermalization takes place.

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Kondo Lattice: Real-Space Renormalization-Group Approach

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A one-dimensional analog of the Kondo lattice is studied by a renormalization-group technique. Previous mean-field results are shown to be reasonable: The system undergoes a second-order crossover transition from an antiferromagnetic state to a Kondo-like state at zero temperature as the spin–conduction-electron coupling is increased. Estimates are given for critical exponents and the behavior of correlation functions near the transition.

$CeAl_2$, $CeAl_3$, and many rare-earth compounds[1] behave anomalously at low temperature—either they do not order magnetically or they have a very low transition temperature. Several authors[1-3] attribute this to a Kondo effect. While the Kondo problem is now solved[4] for a single magnetic impurity, the Kondo-lattice problem which involves one impurity per cell remains an open question. In order to investigate the general properties of this kind of system, Doniach[5] has previously introduced a simple one-dimensional (1D) analog Hamiltonian, the "Kondo necklace":

$$H = J\sum_i \vec{S}_i \cdot \vec{\tau}_i + W\sum_i (\tau_i^x \tau_{i+1}^x + \tau_i^y \tau_{i+1}^y), \qquad (1)$$

where J is positive and where $\vec{S}_i$ and $\vec{\tau}_i$ are two independent sets of Pauli operators [$\tau_i^x = \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$, etc.].

In this Hamiltonian the 1D electron gas has been replaced[5,8] by a set of pseudo spins $\frac{1}{2}\vec{\tau}_i$ regularly spaced on an infinite linear lattice.

The general qualitative behavior of this model may be seen from the weak- and strong-coupling limits: $J/W = 0$ and ∞. For $J = 0$, it reduces to the X-Y quantum-spin chain,[7] which is equivalent to a 1D spinless Fermi gas with one electron per atom.[6-8] For small J/W, we expect that the S spins couple antiferromagnetically via the τ spins, leading to a ground state of broken symmetry exhibiting characteristic spin-wave–like excitations. On the other hand, when $J/W = \infty$, the system reduces to a set of noninteracting singlet-triplet cells leading to a singlet ground state for (1) well separated by a gap of $4J$ from the first excited states. So, we expect a transition from a magnetic behavior to a Kondo-like behavior by increasing the ratio J/W. Within a mean-field approximation,[5,9] this transition occurs at $(J/W)_c = 1$. But it is well known that fluctuation effects, in one dimension, strongly affect the mean-field results. The purpose of this Letter is to present a renormalization-group approach to the problem. Our results confirm the general qualitative features of the model previously suggested.[5,9] We find a critical value $(J/W)_c \cong 0.4$ at which there is a second-order transition at zero temperature in which the system crosses over from antiferromagnetic to Kondo-like behavior. The critical indices for this fixed point are estimated and found to be similar (but not identical) to those for an Ising chain in a transverse field. The approach appears to be generalizable to other quantum spin systems.

The method that we use was proposed to us by Scalapino,[10] who originally used it for the Ising chain in a transverse field. This method is also used in some high-energy problems.[11] The approach is to divide the chain into equal blocks. The Hamiltonian for one block is solved exactly. We then rewrite the interaction between adjacent blocks in terms of single-block eigenstates. We then consider a new block formed with two adjacent old blocks and so on. The approximation consists in selecting only a subset of levels—the lowest if possible—at each step of the iteration. Our calculations consist of two slightly different versions of this general scheme. A brief outline of these methods is presented here; further details will be given elsewhere.[12]

The first method deals with initial blocks of two sites and considers four levels at each iteration. We replace a block of two sites (four spins) by a new set of two spins $\vec{S}$ and $\vec{\tau}$ while maintaining the form of the Hamiltonian. This is made possible by considering, at step n, the following expression for the Hamiltonian:

$$H^{(n)}=\tfrac{1}{2}\{W_1^{(n)}\textstyle\sum_i(\tau_i^{+(n)}\tau_{i+1}^{-(n)}+\tau_i^{-(n)}\tau_{i+1}^{+(n)})+W_2^{(n)}\sum_i(\tau_i^{+(n)}S_{i+1}^{-(n)}+\tau_i^{-(n)}S_{i+1}^{+(n)})$$
$$+W_3^{(n)}\textstyle\sum_i(S_i^{+(n)}\tau_{i+1}^{-(n)}+S_i^{-(n)}\tau_{i+1}^{+(n)})+W_4^{(n)}\sum_i(S_i^{+(n)}S_{i+1}^{-(n)}+S_i^{-(n)}S_{i+1}^{+(n)})\}$$
$$+J_{\parallel}^{(n)}\textstyle\sum_i S_i^{z(n)}\tau_i^{z(n)}+\frac{1}{2}J_{\perp}^{(n)}\sum_i(S_i^{+(n)}\tau_i^{-(n)}+S_i^{-(n)}\tau_i^{+(n)}),\qquad(2)$$

where $\tau_i^{\pm}=\tau_i^x\pm i\tau_i^y$ and similarly for S_i.

Initially we set $W_1^{(0)}=W$, $W_2^{(0)}=W_3^{(0)}=W_4^{(0)}=0$, and $J_{\parallel}^{(0)}=J_{\perp}^{(0)}=J$, and we identify $\vec{S}_i^{(0)}$ and $\vec{\tau}_i^{(0)}$ with the original spins entering (1). Then, at each step, we solve the 16×16 matrix representing the Hamiltonian for a block j of two sites $(1,j)$ and $(2,j)$. This matrix splits into blocks according to the possible eigenvalues of the z projection of the total spin: $\Sigma^{z(n)}=\tau_1^{z(n)}+\tau_2^{z(n)}+S_1^{z(n)}+S_2^{z(n)}$. We consider the two lowest states $s^{(n+1)}$ and $t_0^{(n+1)}$ of energies $E_s^{(n+1)}$ and $E_0^{(n+1)}$ of the 6×6 matrix corresponding to $\Sigma^{z(n)}=0$, and the ground states $t_+^{(n+1)}$ and $t_-^{(n+1)}$ of energy $E_+^{(n+1)}=E_-^{(n+1)}$ of the two 4×4 spin-degenerate matrices corresponding to $\Sigma^{z(n)}=\pm1$. We then identify the four states $s^{(n+1)}$, $t_0^{(n+1)}$, $t_+^{(n+1)}$, and $t_-^{(n+1)}$ with those of two new spins $\vec{S}_j^{(n+1)}$ and $\vec{\tau}_j^{(n+1)}$ (one set per block) by setting[13]

$$J_{\parallel}^{(n+1)}=\tfrac{1}{4}(E_+^{(n+1)}+E_-^{(n+1)}-E_0^{(n+1)}-E_s^{(n+1)}),$$
$$J_{\perp}^{(n+1)}=\tfrac{1}{4}(E_0^{(n+1)}-E_s^{(n+1)}).\qquad(3)$$

Then, by taking the matrix elements of $S_{1,j}^{\pm(n)}$, $S_{2,j}^{\pm(n)}$, $\tau_{1,j}^{\pm(n)}$, and $\tau_{2,j}^{\pm(n)}$ between these four states, we find that these old operators can be expressed as linear combinations of the new ones $S_j^{+(n+1)}$ and $\tau_j^{+(n+1)}$ so that

$$\begin{pmatrix}\tau_{1,j}^{+}\\ S_{1,j}^{+}\end{pmatrix}^{(n)}=\gamma_L^{(n+1)}\begin{pmatrix}\tau_j^{+}\\ S_j^{+}\end{pmatrix}^{(n+1)},$$
$$\begin{pmatrix}\tau_{2,j}^{+}\\ S_{2,j}^{+}\end{pmatrix}^{(n)}=\gamma_R^{(n+1)}\begin{pmatrix}\tau_j^{+}\\ S_j^{+}\end{pmatrix}^{(n+1)},\qquad(4)$$

where $\gamma_L^{(n+1)}$ and $\gamma_R^{(n+1)}$ are 2×2 matrices which can be determined at each step n. The same equations hold for τ^- and S^-. Formulas (4) allow us to express the original coupling between adjacent blocks j and $j+1$ in terms of the new spins $\vec{S}_j$, $\vec{\tau}_j$, $\vec{S}_{j+1}$, and $\vec{\tau}_{j+1}$. We recover exactly the same form as in (2) with new parameters $W_i^{(n+1)}$; and we get the following recursion relations:

$$\begin{pmatrix}W_1\\ W_2\\ W_3\\ W_4\end{pmatrix}^{(n+1)}=\tilde{R}^{(n+1)}\begin{pmatrix}W_1\\ W_2\\ W_3\\ W_4\end{pmatrix}^{(n)},\qquad(5)$$

where $\tilde{R}^{(n+1)}$ is the 4×4 matrix transpose of $R^{(n+1)}=\gamma_L^{(n+1)}\otimes\gamma_R^{(n+1)}$ (with $\otimes$ denoting the tensorial product).

This first method also allows a calculation of the correlation functions at $T=0$. Suppose we stop the iteration scheme at step n. The ground state of the original finite chain of 2^n sites is then approximated by the state $s^{(n)}$ of two spins $S_1^{(n)}$ and $\tau_1^{(n)}$. We can define end-to-end correlation functions for this finite chain by

$$g_{\tau\tau}^{(n)}=\tfrac{1}{2}\langle\tau_1^{+(0)}\tau_{2^n}^{-(0)}\rangle,\quad g_{S\tau}^{(n)}=\tfrac{1}{2}\langle S_1^{+(0)}\tau_{2^n}^{-(0)}\rangle,$$
$$g_{\tau S}^{(n)}=\tfrac{1}{2}\langle\tau_1^{+(0)}S_{2^n}^{-(0)}\rangle,\quad g_{SS}^{(n)}=\tfrac{1}{2}\langle S_1^{+(0)}S_{2^n}^{-(0)}\rangle,\qquad(6)$$

where the brackets take the mean value in the $s^{(n)}$ ground state. By an iterative use of (4), we get immediately

$$\begin{pmatrix}g_{\tau\tau}\\ g_{\tau S}\\ g_{S\tau}\\ g_{SS}\end{pmatrix}^{(0)}=P^{(n)}\begin{pmatrix}1\\ -1\\ -1\\ 1\end{pmatrix},\qquad(7)$$
$$P^{(n)}=R^{(1)}R^{(2)}\cdots R^{(n)}.$$

At each step we check that $g_{S\tau}^{\,2}=g_{SS}g_{\tau\tau}$ and that g_{SS} and $g_{\tau\tau}$ are negative while $g_{S\tau}=g_{\tau S}$ is posi-

tive.

The parameters $W_i^{(n)}$, $J^{(n)}$, and $J^{(n)}$ and the correlation functions are studied when $n \to \infty$ from (3), (5), and (7). Two different types of behavior are observed, depending on the ratio J/W. When $J/W < 0.411$, one of the $W_i^{(n)}$ tends to a nonzero value while all the other parameters tend to zero—we recover in this case a broken-symmetry ground state. The correlation functions tend to constant values which have been plotted as a function of J/W in Fig. 1. In this magnetic region, $-g_{SS}^{\infty}$ and $-g_{\tau\tau}^{\infty}$ can be interpreted as the square of the S and τ magnetization.[14] When $J/W > 0.411$, all the $W_i^{(n)}$—as well as $J_{\parallel}^{(n)}$—tend to zero, while $J_{\perp}^{(n)}$ tends to a nonzero value, leading to a singlet ground state separated from the first excited states by a gap $\Delta = 2(J_{\parallel}^{\infty} + J_{\perp}^{\infty}) \neq 0$. The plot of Δ as a function of J/W is given by curve 1 in Fig. 2. In this range, the correlation functions tend to zero as 2^{-n} when $n \to \infty$ (this appears to correspond to a $1/x$ behavior, where x denotes the distance) and we have plotted

$$X_{SS} = \lim_{n\to\infty} 2^n g_{SS}^{(n)}, \quad X_{S\tau} = \lim_{n\to\infty} 2^n g_{S\tau}^{(n)},$$

and

$$X_{\tau\tau} = \lim_{n\to\infty} 2^n g_{\tau\tau}^{(n)},$$

as a function of J/W in Fig. 1. Before and after the transition we can define the critical indices β for the magnetization, t for the correlation functions, and s for the gap so that

$$-g_{SS}^{\infty} \propto -g_{\tau\tau}^{\infty} \propto g_{\tau S}^{\infty} \propto [(J/W)_c - J/W]^{2\beta},$$
$$-X_{SS}^{\infty} \propto -X_{\tau\tau}^{\infty} \propto X_{\tau S}^{\infty} \propto [J/W - (J/W)_c]^{-t}, \quad (8)$$
$$\Delta \propto [J/W - (J/W)_c]^{s}.$$

We obtain $\beta = 0.4$, $t = 0.7$, and $s = 1$ with 5% error [mean-field theory gives $\beta = 0.5$ (Ref. 5)].

It is apparent that this first method is highly approximate since we retain only four levels at each step. This is a strong limitation since these four states as defined have no reason to be the four lowest states of the total set of sixteen levels for one block. For $J/W > 0.2$, there is an extra doublet crossing over the higher singlet level after a few sets of iterations. For this reason, we have applied another more systematic method which deals with any number of levels n_L in such a way that they always remain the lowest.

In the second method, we do not try to rewrite the Hamiltonian as a spin Hamiltonian but generalize to the following form at each step n:

$$H^{(n)} = \sum_i D_i^{(n)} + \sum_i (A_i^{(n)} \tilde{B}_{i+1}^{(n)} + \tilde{A}_i^{(n)} B_{i+1}^{(n)}), \quad (9)$$

where in a basis $|l\rangle^{(n)}$ attached to each site $l = (1, 2, \ldots, n_L)$, $D^{(n)}$ is an $n_L \times n_L$ diagonal matrix of elements $E_l^{(n)}$, and where $A^{(n)}$ and $B^{(n)}$ are $n_L \times n_L$ nonsymmetric real matrices. As before, we

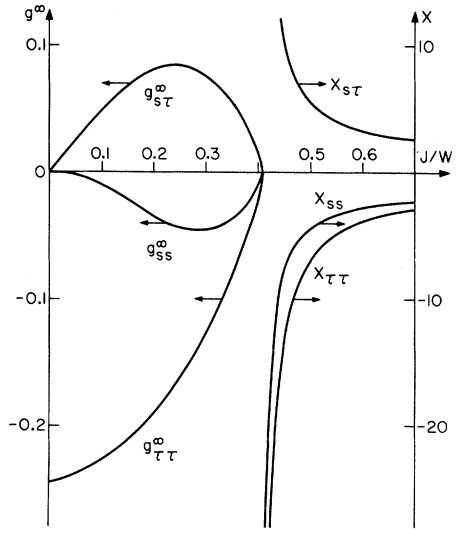

FIG. 1. Plot of g^{∞} (before the transition) and $X = \lim_{n\to\infty}(2^n g^{(n)})$ (after the transition) as a function of J/W.

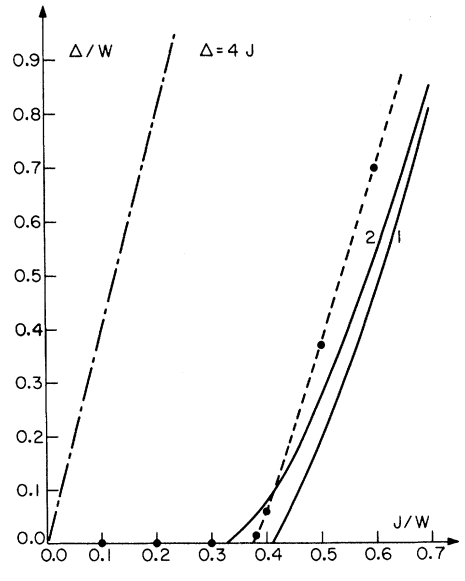

FIG. 2. Plot of the gap Δ as a function of J/W. Curve 1 corresponds to the first method, curve 2 to the second method with $n_L = 4$ and a vanishing anisotropy; and the dots correspond to the second method with $n_L = 16$.

diagonalize the $n_L^2 \times n_L^2$ matrix representing the Hamiltonian for one block of two sites and obtain the representation of the interblock coupling in terms of the eigenstates of the blocks. The approximation consists in restricting to the subspace of lowest energies. This method can be handled easily by computer and we have written a subroutine which transforms the set $\{D,A,B\}$ into a new set of dimension n_L. We have done the calculations for both $n_L = 4$ and $n_L = 16$.

For $n_L = 4$, we recover the same results as with the first method in the range $J/W < 0.2$ where we use exactly the same set of levels. In the range $J/W > 0.2$, after a few steps of iteration of doublet appears instead of a singlet at the top of the set of lowest levels. To avoid ambiguity in the choice of one member of the doublet, a small asymmetry term $\epsilon J \sum_i (S_i^x \tau_i^x - S_i^y \tau_i^y)$ has been added to the original Hamiltonian and the results are studied as a function of ϵ for small ϵ.

With this second method, we observe that all the elements of $A^{(n)}$ and $B^{(n)}$ tend to zero above the transition, while some of them tend to a nonzero value below the transition. The results for the gap

$$\Delta = \lim_{n\to\infty} [E_2^{(n)} - E_1^{(n)}]$$

are shown in Fig. 2. We find that $(J/W)_c = 0.325$ for $n_L = 4$ and a vanishing anisotropy, and that $(J/W)_c = 0.375$ for $n_L = 16$.

It is fruitful to compare these results with those of the one-impurity Kondo problem.[4] Let us define a dimensionless parameter $g^{(n)}$ at each step of the iteration:

$$g^{(n)} = J_\perp^{(n)} / \max|W_i^{(n)}| ,$$

in the first method; or

$$g^{(n)} = (E_2^{(n)} - E_1^{(n)}) / \max|a_{ij}^{(n)} b_{kl}^{(n)}| ,$$

(where $a_{ij}^{(n)}$ and $b_{ij}^{(n)}$ are the elements of $A^{(n)}$ and $B^{(n)}$) in the second method. This parameter plays the same role as $g = J\rho$, where ρ is the density of states for the conduction electrons, in the Wilson case.[4] In the one-impurity case, g always tends to infinity. Except for $J = 0$, the fixed point $g^* = 0$ is never reached.

For the Kondo lattice, on the other hand, we have shown that the fixed point $g^* = 0$ is reached for $J < J_c$, while another Kondo-like fixed point $g^* = \infty$ is reached for $J > J_c$. The change of behavior at $J = J_c$ corresponds to the usual definition of a crossover effect.

It is also useful to contrast these results with the properties of the Ising chain in a transverse magnetic field where the same kind of crossover exists: By increasing the magnetic field, a transition from a degenerate ground state to a singlet ground state arises, exactly as in our case. The exact critical exponents of the transverse Ising models are $\beta = 0.125$, $t = 0.75$, and $s = 1$,[15] in contrast to the Kondo-lattice results reported above.

In conclusion, we remark that the system studied here is a special case of a coupled two-chain problem. A systematic use of this method to study the general coupled two-chain problem is in progress.

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[14]For $J = 0$ we find $g_{TT} = -0.244$. The exact value (Ref. 6) for the X-Y model is $g_{TT} = 0.636$ for the short-range order and zero for the long-range order.

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