Theory of the Off-Shell Pion-Nucleon t Matrix*

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A theory of the off-shell πN t matrix is proposed. The theory, which employs dispersion-theoretic techniques, incorporates the on-shell information, including the inelasticities, and derives the additional information required to continue off shell from the underlying field theory. Results are presented for the P-wave states and are compared with those of other approaches.

Now that the determination of the π-nucleus scattering using the multiple-scattering theories has reached a fair degree of precision, it is becoming more desirable to take realistic off-shell pion-nucleon (πN) t matrices as input. Two different procedures have been used to obtain these. Phenomenological approaches, based on potential theory, have been proposed.[1,2] The most realistic of these assumes a distinct separable interaction in each partial-wave state whose form is determined directly from the on-shell data. The limitations of these approaches have led to the more realistic field-theoretic methods, which construct the off-shell t matrix starting from the underlying field-theoretic description of the πN interaction.[3-5] To make progress, however, one is usually forced to make the static approximation.

One of the main shortcomings of the field-theoretic methods for determining the on-shell and fully off-shell (FOS) πN t matrices is that the form factor, which enters through the interaction Lagrangian, is directly related to the half-shell πN t matrix (HST).[3] This factorization of the HST, which is a direct consequence of the static limit, leads to separable amplitudes.[6] Now since the form factor must be the same in all states of the πN system, it follows that the HST must have the same state dependence as the on-shell t matrix; or, equivalently, that the half-shell function[7] (HSF) must be state independent. This is a rather unrealistic constraint and is also inconsistent with the results of the potential-theoretic approaches. In addition to this difficulty, the field-theoretic approaches usually make the one-meson approximation and also ignore the effects of the inelasticities in the determination of the form factor. This prevents one from properly taking account of the inelasticities in the determination of the on-shell and FOS t matrices.[5] Attempts have been made to obtain the form factor, including the effects of the inelasticities, by inverting the field-theoretic equations, but this then leads to state-dependent form factors.[5]

Motivated by these considerations, I present here a theory of the πN HST which can be thought of as combining the best features of the two above approaches. It has the character of the potential-theoretic approach, i.e., it utilizes the on-shell information, correctly including the inelasticities, at all energies, but without invoking the separable constraint. On the other hand, the additional information needed to continue off shell is obtained by appealing in a natural and consistent way to the underlying field theory, without the necessity of resorting to the static limit.

Among the basic features that a theory of the πN HST should possess, I believe that one can list the following: (1) covariance, (2) relativistic kinematics, (3) crossing symmetry, (4) nonstatic interactions, and (5) a field-theoretic foundation. I propose that a theory possessing these attributes can be formulated by using the dispersion-theoretic methods applied previously to the NN HST.[8] To accomplish this, two ingredients are required: first, a specification of the off-shell dynamics; and second, a dynamical equation.

In view of the above desired features, I am led to define[9] the half-off-shell dynamics as illustrated in Fig. 1, describing the process in which a π and a N come in on their mass shells with rela-

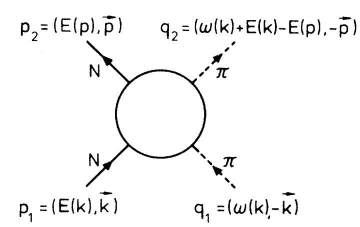

FIG. 1. The specification of the half-off-shell dynamics. The four-momenta are given in the c.m. system.

tive momentum $\vec{k}$ in the center-of-mass system, and after the interaction go out with relative momentum $\vec{p}$, the N remaining on mass shell but the final π going off mass shell in order to conserve energy. On shell, when $|\vec{p}|\to|\vec{k}|$, this reduces to the usual on-mass-shell amplitude.

The most general form of the off-shell πN transition amplitude consistent with covariance and my choice of the off-shell dynamics can be decomposed in terms of the two invariant off-shell amplitudes A and B as[10]

$$T(p_2, q_2; p_1, q_1) = \bar{u}(p_2)[A(p_2, q_2; p_1, q_1) + \not{Q} B(p_2, q_2; p_1, q_1)]u(p_1), \tag{1}$$

where $\not{Q} = \frac{1}{2}(\not{q}_1 + \not{q}_2)$ and $\bar{u}(p_2)$ and $u(p_1)$ are the four-component spinors of the nucleon. (We ignore isospin to simplify the presentation.) The half-shell partial-wave amplitudes that we work with are defined by

$$h_{l\pm}(p, w_k) = \tfrac{1}{2}\int_{-1}^{+1} dz[M_1 P_l(z) + M_2 P_{l\pm1}(z)], \tag{2}$$

where $l\pm = (J \mp \frac{1}{2})\pm$, $w_k = \sqrt{s} - M$, s is the c.m. energy, M is the nucleon mass, and the $P_l(z)$ are the Legendre polynomials. Equation (2) explicitly reveals the on- and off-shell energy and momentum dependences, and the amplitudes M_1 and M_2 are given in terms of the invariant amplitudes by

$$M_1 = \tfrac{1}{2}[A + (\sqrt{s} - M)B],$$
$$M_2 = \tfrac{1}{2}pk[-A + (\sqrt{s} + M)B]/W(p)W(k), \tag{3}$$

where $W(k) = E(k) + M$. In the on-shell limit, we have

$$h_{l\pm}(w_k) = \frac{\sqrt{s}\{\eta_{l\pm}(w_k)\exp[2i\Delta_{l\pm}(w_k)] - 1\}}{2ikW(k)}, \tag{4}$$

where $\eta_{l\pm}$ and $\Delta_{l\pm}$ are the inelasticity parameter and the real part of the phase shift, respectively.

We now impose unitarity on the HST, taking care to account properly for the inelasticities. Restricting from now on our consideration to the P-wave states and explicitly introducing the isospin, we assume that half-shell unitarity is of the form[2]

$$\mathrm{Im}\hat{h}_\alpha(p, w_k) = \exp[-i\varphi_\alpha(w_k)]\sin\varphi_\alpha(w_k)\hat{h}_\alpha(p, w_k), \tag{5}$$

where

$$\varphi_\alpha(w_k) = \tan^{-1}\left(\frac{1 - \eta_\alpha(w_k)\cos2\Delta_\alpha(w_k)}{\eta_\alpha(w_k)\sin2\Delta_\alpha(w_k)}\right).$$

The reduced amplitudes, $\hat{h}_\alpha(p, w_k) = h_\alpha(p, w_k)/pk$, are introduced in order to remove the kinematical singularities and the label α to indicate the four possible P-wave states: $\alpha = P11$, $P13$, $P31$, $P33$.

The next feature that I wish to incorporate into the theory is crossing symmetry. We make the approximation of assuming that the crossing symmetry relation for the πN HST is of the same form as in the static case, i.e.,

$$\hat{h}_\alpha(p, -w_k) = \textstyle\sum_\beta A_{\alpha\beta}\hat{h}_\beta(p, w_k), \tag{6}$$

where $A_{\alpha\beta}$ is the 4×4 crossing matrix as given in Ref. 3. The main reason for doing this is to contain the size of the problem, since in this case only P waves cross into the P-wave state.

Unitarity and crossing symmetry specify the basic analytic structure which the partial-wave πN HST must possess, consisting of a right-hand cut beginning at $+\mu$ (μ is the π mass) and a left-hand cut beginning at $-\mu$ in the w_k plane. Assuming this analyticity, we postulate the following fixed-p, subtracted dispersion representation for the reduced πN HST,[11]

$$\hat{h}_\alpha(p, w_k) = \hat{h}_\alpha(w_p) + \hat{B}_\alpha(p, w_k) + \frac{w_k - w_p}{\pi}\int_\mu^\infty dw_q \frac{\mathrm{Im}\hat{h}_\alpha(p, w_q)}{(w_q - w_p)(w_q - w_k)} + \frac{w_k - w_p}{\pi}\int_{-\infty}^{-\mu} dw_q \frac{\mathrm{Im}\hat{h}_\alpha(p, w_q)}{(w_q - w_p)(w_q - w_k)}. \tag{7}$$

In (7), $\hat{h}_\alpha(w_p)$ is the reduced on-shell t matrix, the third term on the right-hand side is the contribution from the unitarity cut, the fourth term is the contribution from the crossed cut, and $\hat{B}_\alpha(p, w_k)$, which includes the Born singularities, is related to the off-shell Born amplitudes.

Using (5) and (6), one finds that (7) becomes a coupled, singular, linear integral equation for $\hat{h}_\alpha(p, w_k)$ which can be reduced to the following coupled integral equation of Fredholm form[12]:

$$h_\alpha(p, w_k) = \tau_\alpha(p, w_k) + \sum_\beta \int_\mu^\infty dw_q K_{\alpha\beta}(p, w_q, w_k) k h_\beta(p, w_q)/q, \tag{8}$$

where we have now returned to the unreduced amplitude, and where

$$\tau_\alpha(p, w_k) = kp^{-1} h_\alpha(w_p) \exp[u_\alpha(p, w_k)] + B_\alpha(p, w_k) + \frac{w_k - w_p}{\pi} \exp[u_\alpha(p, w_k)] pk \int_\mu^\infty dw_q \frac{\exp[-u_\alpha(p, w_q) + i\varphi_\alpha(w_q)] \sin\varphi_\alpha(w_q) \hat{B}_\alpha(p, w_q)}{(w_q - w_p)(w_q - w_k)},$$

$$u_\alpha(p, w_k) = \frac{w_k - w_p}{\pi} \int_\mu^\infty dw_q \frac{\varphi_\alpha(w_q)}{(w_q - w_p)(w_q - w_k)},$$

$$K_{\alpha\beta}(p, w_q, w_k) = \frac{A_{\alpha\beta}}{\pi}(w_k - w_p) \frac{\exp[-i\varphi_\beta(w_q)] \sin\varphi_\beta(w_q)}{(w_q + w_p)(w_q + w_k)} \frac{\exp[u_\alpha(p, w_k)]}{\exp[u_\alpha(p, -w_q)]}.$$

In (8), which is the dynamical equation for the πN HST, the inputs are the phase-shift parameters at all energies and the Born or driving term, the latter permitting us to make contact, in a natural and consistent way, with the underlying field theory. The subtracted form of the equations allows maximal use of the on-shell information and, in addition, helps to suppress the high-energy contributions.

The Born term is taken to be the sum of the off-shell Feynman amplitudes corresponding to the exchanges of the N in the s and u channels and the ρ and σ mesons in the t channel. These amplitudes, which are obtained without taking the static limit, are easily calculated.[13] Since the final π is off mass shell, they will involve the πN vertex function—which we take from Nutt and colleagues.[14] We also take account of the finite width of the ρ and σ mesons. For the ρ-exchange vertex and propagator, we use the parametrizations given by Iachello, Jackson, and Lande.[15] We use the same parametrizations for the σ exchange, but with a mass of 993 MeV, a width of 180 MeV, and with couplings determined from the σ model.[16] Furthermore, we introduce dipole form factors, with a cutoff of 1.47 GeV, in the N-exchange amplitudes, and eikonal form factors at the ρNN and σNN vertices. The introduction of these form factors is somewhat unsatisfactory but is necessary to assure reasonably convergent amplitudes. It represents our lack of a complete strong-interaction dynamical theory and, more particularly, our ignorance of the short-range contributions.

The solid curve of Fig. 2, illustrates the HSF,[7] $f_\alpha(p, w_k)$, in the four P-wave states obtained from (8) together with the above Born term. The dashed curve shows the HSF for the separable interaction[2] obtained with the same on-shell phase-shift parametrization, and the dot-dashed curve shows the Kisslinger[1] HSF ($=p/k$). Except for the $P11$ state, the HSF obtained from (8) is qualitatively similar to the separable result. The major contribution comes from the on-shell t matrix and the N exchange—the on-shell information determining the overall form.

The ρ and σ exchanges give a contribution varying from about 20% of the N exchange at small p

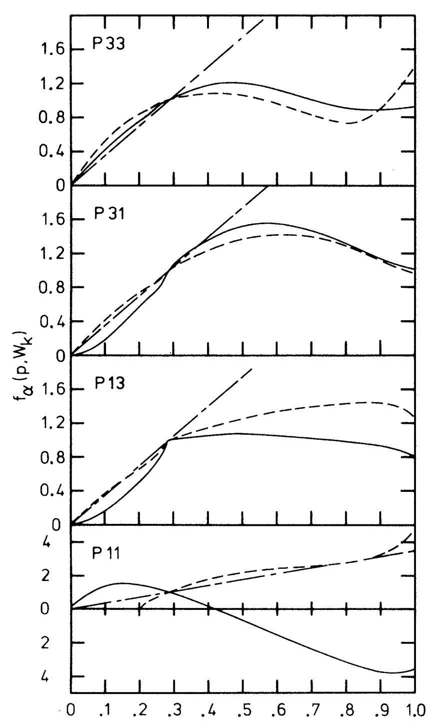

FIG. 2. The half-shell functions at $w_k = 0.363$ GeV for the four P-wave states.

to about 50–60% at large p. The contribution from the crossed cut is about the same order of magnitude. As expected, above $p \simeq 500$ MeV/c the results are rather sensitive to the form factors (this region is also sensitive to the high-energy phase shifts[17]). In the $P11$ state, there is a noticeable difference between the separable result and the present one. This is primarily due to the large contribution from the s-channel nucleon pole. It is worth pointing out that the phase shift changes sign in this state so that the one-term separable interaction, which yields a HSF which cannot change sign, is not really defined. In the present case where this constraint is removed, the $P11$ HSF at this energy does change sign. At different energies, I obtain results for the HSF which are qualitatively similar to these.

In my approach, the FOS πN t matrix is obtained by writing a further subtracted dispersion representation, subtracting at the half-shell point. It involves as input the on-shell information, the HST, and the Born term deduced from the FOS Feynman amplitudes, and hence correctly incorporates the two-body dynamics, including the inelasticities. The determination of this t-matrix, as well as the π-nucleus scattering using these, is in progress and will be published later elsewhere. The HST of this paper also has direct applications in pion absorption.[18]

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