# Nonlinear Evolution of the Collisionless Tearing Mode*

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(Received 22 February 1977)

For a broad $\vec{k}$ spectrum, the collisionless tearing mode grows linearly until the Landau electrons become magnetized by the wave field. The electrons then spatially diffuse as a result of turbulent magnetic-gradient drifts which leads to a nonlinearly enhanced growth rate. At large amplitudes the nonlinear growth rate is dominated by ions.

In a plane, one-dimensional magnetic neutral sheet, the collisionless tearing mode is destabilized by the dissipative Landau resonance of electrons ($j=e$) and ions ($j=i$) which move parallel to the dc magnetic field.[1,2] For a Harris[3] equilibrium with a dc field $\vec{B}_0=B_0\tanh(z/\lambda)\hat{x}$, the resonant particles are spatially localized to a height around $z=0$ given by $|z|\leq d_j=\lambda\epsilon_j^{1/2}$, where $\epsilon_j=R_j/\lambda$ and $R_j$ is the Larmor radius based on the thermal velocity $a_j=(2T_j/m_j)^{1/2}$ and the asymptotic field $B_0$; i.e., $R_j=a_j/\Omega_j$, with $\Omega_j=eB_0/m_jc$ the gyrofrequency. For comparable electron and ion temperatures, the electrons make the dominant contribution to the linear growth rate. The dispersion relation for the wave frequency $\omega=\omega_r+i\gamma$ is given approximately by[1,4]

$$\omega_r=k_yu_e,$$
$$\gamma_e=\pi^{-1/2}ka_e\left[\frac{T_e+T_i}{T_e}\right]\epsilon_e^{3/2}(\cos^2\theta-k^2\lambda^2), \tag{1}$$

where $u_e=\epsilon_e a_e$ is the diamagnetic drift speed and $\theta=\cos^{-1}(k_x/k)$ is the angle the vector $\vec{k}$ makes with the magnetic field direction; $k^2=k_x^2+k_y^2$ with $y$ being the direction of the equilibrium current. Since $\epsilon_e\ll 1$ usually pertains, the Landau-resonant electrons have velocities $v\sim\omega_r/k\sim a_e\epsilon_e\ll a_e$. The tearing mode has an induction electric field $\vec{E}$ which can freely accelerate particles in the region $|z|<d_j$, but only produces and $\vec{E}\times\vec{B}_0$ drift of the plasma for $|z|>d_j$. However, since $ka_j\gg\omega_r$ or $\gamma$, only the Landau-resonant electrons experience an effective dc-electric-field acceleration, and make a nonreactive contribution to the perturbed current.

The first nonlinear theory by Biskamp, Sagdeev, and Schindler[5] was a one-dimensional ($k_y=0$) quasilinear analysis which showed that the mode stabilized by forming a velocity-space plateau in the $v_x$ electron distribution function. The rms saturation amplitude of the wave's $z$ magnetic field component $\overline{B}_1$ was estimated as $\overline{B}_1/B_0\sim\epsilon_e^3$. Galeev and Zelenyi[4] showed that for a two-dimensional $k$ spectrum ($k_y\neq 0$), quasilinear plateau formation did not occur. They argued that stabilization would occur when the electron Larmor radius based on the turbulent normal field component $\overline{B}_1$ satisfied $kR_1\sim 1$, where $R_1=a_e/\overline{\Omega}_1$ and $\overline{\Omega}_1=e\overline{B}_1/m_ec$. At this point electrons begin to execute cyclotron orbits in the wave's normal magnetic field, thus preventing the Landau resonance. Mathematically the Landau resonant denominator $\omega-\vec{k}\cdot\vec{v}$ is replaced by $\omega-n\overline{\Omega}_1$, where $n$ is the cyclotron harmonic number. The electron response to the wave electric field is just an $\vec{E}\times\vec{B}_1$ drift which is nondissipative, and electrons within $|z|<d_e$ are not accelerated in the direction of the wave electric field. From their linear stability calculations with a constant normal component,[6] Galeev and Zelenyi argue that the electron dielectric response to the wave becomes reactive when $kR_1\sim 1$, and makes a strongly stabilizing contribution to the dispersion relation. The condition $kR_1\sim 1$ implies a rms saturation amplitude of $\overline{B}_1/B_0\sim\epsilon_e$ since $k\lambda\lesssim 1$.

In the tearing mode the free energy comes from the macroscopic dc-magnetic-field configuration, and is converted into turbulent wave energy and energization of the resonant particles. The predicted low saturation amplitude implies that very little of the available free energy is dissipated.

Since the tearing mode exists because of Landau dissipation, it is not clear that the growth of a turbulent wave field, which naively increases the dissipation rate, should lead to stabilization of the mode. The theory of strong plasma turbulence[7,8] predicts that particle orbits undergo diffusion in the turbulent wave fields. If the particles are unmagnetized, orbit diffusion broadens the Landau-resonant region of velocity space. For the tearing mode, it will be shown elsewhere that for $\overline{B}_1/B_0<\epsilon_e$ ($kR_1>1$), the orbit diffusion broadening of the electron Landau resonance does not change the linear growth rate in lowest order. Hence the mode will grow to $\overline{B}_1/B_0\sim\epsilon_e$, and the electrons will become magnetized in the turbulent wave fields.

For magnetized particles and low-frequency, long-wavelength modes, the turbulence produces a spatial diffusion because of random $\vec{E}\times\vec{B}$ and magnetic-gradient drift velocities.[9] A general result of the strong-turbulence theory is that the cyclotron denominator $\omega - n\overline{\Omega}_1$ is modified to $\omega - n\overline{\Omega}_1 + i\vec{k}\vec{k}:\overleftrightarrow{D}$, where $\overleftrightarrow{D}$ is the spatial diffusion tensor [see also Eq. (10)]. With the spatial-diffusion modification, the electron dielectric response to the wave does include a dissipative (i.e., imaginary) contribution which can take the place of the previous ($kR_1 > 1$) dissipative Landau-resonant term. Physically, the turbulence allows the particles within $|z| < d_j$ to diffuse spatially across the magnetic field, while the magnetic field, while their small cyclotron radius $kR_1 < 1$ keeps them in a region of approximately constant wave electric field. On the average there is a net diffusion current along $\vec{E}$. The instability should continue to grow beyond the level $\overline{B}_1/B_0 \sim \epsilon_e$ with a growth rate which depends on $\vec{k}\vec{k}:\overleftrightarrow{D}$.

An exact calculation of the particle orbits and the spatial diffusion tensor is quite complicated because of the turbulent spatial variation of the gyrofrequency and localized neutral points in the total magnetic field. Rather than attempt an exact calculation, I develop an *ad hoc*, but physically reasonable model for the regime $\overline{B}_1/B_0 > \epsilon_e$ which shows that the wave-induced turbulent magnetic-gradient drifts lead to a nonlinearly enhanced growth rate. The model should apply to situations where a broad $\vec{k}$ spectrum of tearing modes is excited, and not to cases, such as the tokamak, where geometry constrains the geometry to evolve into a single mode with a large magnetic island.

*Model for diffusing orbits.*—Although the particle orbits in a magnetic neutral sheet are complicated, the usual approximation in analyzing the linear tearing mode is to assume that within the region $d_j$, the particle orbits are straight lines.[2,4] The $x$ and $y$ components of the wave magnetic field vanish at the neutral line and increase only as $\sqrt{\epsilon}z/\lambda$ away from it.[2] Hence the dominant wave component which affects the particle orbits is $B_z$, which is essentially constant across the width $d_j$. The basic physical concept which I wish to examine is the tendency for the electrons to undergo cyclotron motion in the field $B_z$. Therefore I make the reasonable approximation that on the average the electrons within $d_e$ gyrate with a gyrofrequency $\overline{\Omega}_1$ based on the rms $B_z$ field, $\overline{B}_1$.

The field $B_z$, however, is not spatially uniform, but has gradients in the $x$ and $y$ directions which will produce magnetic-gradient drifts. In order to model this drift, I assume that the electron drift velocity is given by

$$\vec{v}_D = -(m_e c v_\perp^2/2e\overline{B}_1^2)\hat{z}\times\nabla B_z. \tag{2}$$

Since the ensemble average of $B_z$, $\langle B_z\rangle$, vanishes, we have $\langle\vec{v}_D\rangle = 0$. The $\vec{E}\times\vec{B}$ drift from the wave electric field is less than $\vec{v}_D$ provided $kR_1 > \epsilon_e$, and I will neglect it. While the guiding-center form for $\vec{v}_D$ is reasonable for the strong-field regions where the local Larmor radius is small, it breaks down in the weak-field regions surrounding local neutral points. However, stochastic scattering of particles at neutral points can only increase the orbit diffusion. Hence the orbit model probably underestimates the spatial diffusion which is produced by the wave turbulence.

*Diffusion tensor.*—With the above model, the electron position as a function of time is

$$\begin{aligned} x - x_0 &= (v_\perp/\overline{\Omega}_1)[\sin(\overline{\Omega}_1 t + \varphi_0) - \sin\varphi_0] + \int_0^t dt'\, v_{D_x}(t'), \\ y - y_0 &= -(v_\perp/\overline{\Omega}_1)[\cos(\overline{\Omega}_1 t + \varphi_0) - \cos\varphi_0] + \int_0^t dt'\, v_{D_y}(t'), \end{aligned} \tag{3}$$

where $\varphi_0$ is the initial ($t=0$) Larmor phase. I denote this trajectory by $\vec{R} = \Delta\vec{R} + \delta\vec{R}$, where $\langle\vec{R}\rangle = \langle\Delta\vec{R}\rangle$ and $\langle\delta\vec{R}\rangle = 0$; $\langle\Delta\vec{R}\rangle$ corresponds to the gyroterms in (3) and $\delta\vec{R}$ to the drift terms.

In order to compute the spatial diffusion tensor $\overleftrightarrow{D}$ and the dielectric function, we must calculate the ensemble average of the orbit function $\exp(-i\vec{k}\cdot\vec{R})$. Using the cumulant expansion,[8] we have $\langle\exp(-i\vec{k}\cdot\vec{R})\rangle \cong \exp[-i\langle\vec{k}\cdot\Delta\vec{R}\rangle - \frac{1}{2}\langle(\vec{k}\cdot\delta\vec{R})^2\rangle]$. The standard diffusion approximation[9] then yields

$$\langle(\vec{k}\cdot\delta\vec{R})^2\rangle = 2t\vec{k}\vec{k}:\overleftrightarrow{D}, \quad \overleftrightarrow{D} = (mcv_\perp^2/2e\overline{B}_1^2)^2\sum_{\vec{k}}(|B_{\vec{k}}|^2/\vec{k}\vec{k}:\overleftrightarrow{D})(\hat{z}\times\vec{k})(\hat{z}\times\vec{k}), \tag{4}$$

where $B_z$ is expressed as a Fourier series, and $\langle B_{\vec{k}}B_{\vec{k}'}\rangle = |B_{\vec{k}}|^2\delta_{\vec{k},\vec{k}'}$ is assumed. In calculating $\overleftrightarrow{D}$, I assumed $kv_\perp/\overline{\Omega}_1 \ll 1$ and $\vec{k}\vec{k}:\overleftrightarrow{D} > \omega \sim \epsilon_e k_y a_e$ which, as is easily checked *a posteriori*, requires $kR_1 > \epsilon_e$.

Since the tearing-mode growth depends only on $\cos^2\theta$, a broad angular $k$ spectrum should be excited;

note that if $k_y=0$, then $\overleftrightarrow{D}=0$. An immediate consequence of the $\vec{k}$ symmetry is $D_{xy}=D_{yx}=0$, from which it follows that

$$D_{xx}D_{yy}=\tfrac{1}{2}(mcv_\perp^2/2e\overline{B}_1^2)^2\textstyle\sum_{\vec{k}}|B_{\vec{k}}|^2. \tag{5}$$

For a reasonably broad angular $k$ spectrum, $D_{xx}$ should be comparable to $D_{yy}$. Hence as a rough approximation we take $D_{xx}\approx D_{yy}$ so that

$$D_{xx}\approx D_{yy}\approx(1/2\sqrt{2})v_\perp^2/\overline{\Omega}_1, \tag{6}$$

where I have defined $\overline{B}_1^2=\sum_{\vec{k}}|B_{\vec{k}}|^2$.

*Tearing-mode dispersion relation.*—Following the original analysis of Laval, Pellat, and Vuillemin,[1] an approximate form of the tearing-mode dispersion relation is given by

$$(2/\lambda)(\cos^2\theta-k^2\lambda^2)$$
$$=-i\textstyle\sum_j(\omega_{pj}^2d_j/c^2T_j)[(\omega-k_yu_j)/k^2]\int d^3v f_{0j}(k_xv_y-k_yv_x)\int_0^\infty d\tau\,\langle[k_xv_y(\tau)-k_yv_x(\tau)]e^{-i\vec{k}\cdot\vec{R}}\rangle e^{i\omega\tau}, \tag{7}$$

where $f_{0j}$ is a Maxwellian distribution with temperature $T_j$ and $\omega_{pj}=(4\pi ne^2/m_j)^{1/2}$ is the plasma frequency based on the density $n$ at the neutral line. The appearance of the ensemble average indicates that (7) is the nonlinear dispersion relation of strong-turbulence theory.[7,8] For simplicity I retain only the cyclotron terms in $v_y(\tau)$ and $v_x(\tau)$; the drift corrections give a contribution which has the same sign as the cyclotron contribution, but is small for a broad angular $\vec{k}$ spectrum. For the ion contribution I assume unperturbed straight-line ion orbits and retain only the Landau-resonant part of the velocity integral. The result is

$$\frac{2}{\lambda}(\cos^2\theta-k^2\lambda^2)=-i\frac{\omega_{pe}^2d_e}{c^2T_e}(\omega-k_yu_e)2\pi m_e\int_0^\infty dv_\perp v_\perp^3f_{0e}\sum_{n=-\infty}^{\infty}\frac{iJ_n'^2}{\omega-n\overline{\Omega}_1+i\vec{k}\vec{k}:\overleftrightarrow{D}}-i\frac{\omega_{pi}^2d_i}{c^2}\sqrt{\pi}\,\frac{\omega-k_yu_i}{ka_i}, \tag{8}$$

where $J_n'$ is the derivative of $J_n$ with respect to its argument, and $f_{0e}$ has been integrated over $v_z$.

For the moment I neglect the ion term, which is small, and consider only the electron term. Suppose we neglect the turbulent gradient drift correction $\vec{k}\vec{k}:\overleftrightarrow{D}$ to the denominator in (8); we then would have the situation discussed by Galeev and Zelenyi[4] where only the electron gyromotion in the turbulent $B_z$ field was considered. For $\omega\approx k_yu_e<\overline{\Omega}_1$ or $k_ya_e/\overline{\Omega}_1<1/\epsilon_e$, only the $n=0$ term contributes. The whole electron term is now real, and the electrons have only a reactive response to the wave. The solution to (8) is $\omega\simeq k_yu_e$ and only stable oscillations result.

The turbulent gradient drift correction, however, allows the electrons to have a dissipative response to the wave. If $\vec{k}\vec{k}:\overleftrightarrow{D}>\omega$, the growth rate obtained from (8) is inversely proportional to a velocity-weighted integral over $(\vec{k}\vec{k}:\overleftrightarrow{D})^{-1}$. Hence, regardless of the precise form for the diffusion tensor, the inclusion of spatial diffusion in the particle orbits leads to tearing-mode growth.

With use of approximation (6) obtained from the model orbits and neglect of $\omega$ in the denominator, the dispersion relation reduces to

$$\omega=k_yu_e+i\sqrt{2}\,\overline{\Omega}_1\left[\frac{T_e+T_i}{T_e}\right]\epsilon_e^{3/2}(\cos^2\theta-k^2\lambda^2). \tag{9}$$

Upon comparing (9) and (1), which is valid for $kR_1>1$, the factor $ka_e$ in (1) is simply replaced by $\overline{\Omega}_1$ when $kR_1<1$ or $\overline{B}_1/B_0>\epsilon_e$. The growth rate is now a function of the rms turbulent wave amplitude so that the growth rate increases with the turbulence level; such behavior is often termed an explosive instability. I conclude that the tearing mode will not saturate at $\overline{B}_1/B_0\sim\epsilon_e$, but will continue to grow at a nonlinearly enhanced rate.

*Transition to ion tearing.*—As the rms amplitude increases above $\overline{B}_1/B_0\sim\epsilon_e$, the electron contribution to the dispersion relation decreases as $1/\overline{\Omega}_1$. The electron term becomes comparable to the ion term when

$$\overline{B}_1/B_0\sim k\lambda\epsilon_i(T_em_e/T_im_i)^{1/4}. \tag{10}$$

For higher amplitudes, the tearing mode is driven by the ion Landau resonance with a growth given by (1) with subscript $e$ replaced by $i$. For $\overline{B}_1/B_0>\epsilon_i$, the ions become magnetized and undergo turbulent magnetic-gradient drifts. The growth rate is now given by (9) with $e\to i$ and $\overline{\Omega}_1=e\overline{B}_1/m_ic$. The behavior of the growth rate as a function of $\overline{B}_1/B_0$ is sketched in Fig. 1.

*Nonlinear evolution.*—Even though the *ad hoc* model for the particle orbits is undoubtedly a gross oversimplification, it is probably reasonable to conclude that the tearing mode will not be

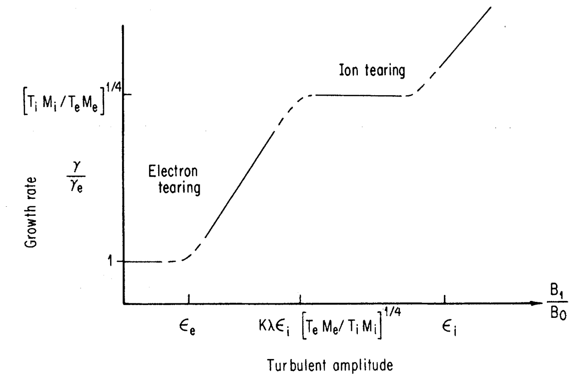


FIG. 1. A sketch of the nonlinear growth rate $\gamma$ of the collisionless tearing mode normalized to the linear electron growth rate $\gamma_e$ [Eq. (1)] as a function of the rms amplitude of the wave's normal magnetic field $\overline{B}_1$ normalized to the external dc field strength $B_0$. The solid part of the curve represents regions of $\overline{B}_1/B_0$ where the calculated growth rates are accurate; the dashed part of the curve represents an extrapolation between parameter regimes.

stabilized by turbulent-wave modifications in the resonant particle dynamics. On the contrary, the plasma dissipation and the tearing-mode growth increase with the turbulence level. Although I have not solved for the final nonlinear temporal evolution of the tearing mode, I can offer the following speculative scenario.

As the rms amplitude $\overline{B}_1$ increases above $\epsilon_i B_0$, the turbulence will begin to modify the macroscopic current profile which is the ultimate source of free energy for the instability. The plasma current density will diffuse outward in $z$ because of turbulent $\vec{E} \times \vec{B}_0$ drifts from the wave induction electric field so that the magnetic-gradient scale length $\lambda$ increases. Hence stored magnetic energy from outside the neutral field region is eventually dissipated and converted into turbulent magnetic energy and plasma heat. The evolution of the magnetic field should resemble the classical resistive diffusion of field lines into the neutral sheet. As $\lambda$ increases, initially unstable modes with $k\lambda < 1$ will be damped by the plasma as $k\lambda$ becomes greater than unity. Hence the energy of all but the longest-wavelength modes will be dissipated into plasma heat.

It is a pleasure to acknowledge many beneficial discussions with Dr. J. Drake, Dr. C. F. Kennel, and Dr. Y. C. Lee.

*This work was partially supported by the National Aeronautics and Space Administration Grant No. NGL-05007-190-54 and by the National Science Foundation Grant No. GA-34148.

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