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# Lattice Gauge Theory, String Model, and Hadron Spectrum

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In Wilson's lattice gauge theory, we obtain an effective Lagrangian for a string with quarks attached to the ends. A set of classical solutions are obtained, and quantized by the Bohr-Sommerfeld method to give the Regge trajectories. The ground-state spectra are those of the SU(6) model in the classical limit.

Wilson's[1] lattice gauge theory is considered to be a promising model for hadrons which provides both quark confinement and asymptotic freedom. As was shown in his original article, the confinement of quarks is achieved by the area law in the strong-coupling approach. The study of hadron spectrum in this model seems to be an urgent problem.[2] In this Letter, we apply the area law to the model including the quark interactions, and obtain an effective Lagrangian which governs the motion of the string made by links and quarks.

Let $X^\mu(\sigma,\tau)$ be a point on the world sheet swept out by the string. The propagation of the lattice-gauge string, whose length is larger than the lattice distance $a$, can be given by the path integration

$$P=\int \exp(iI_S)[\langle \exp(iI_q)\psi_1(X_1(T))\bar{\psi}_2(X_2(T))\psi_2(X_2(0))\bar{\psi}_1(X_1(0))\rangle\langle\exp(iI_q)\rangle^{-1}]\mathfrak{D}X. \tag{1}$$

where

$$I_S=\iint L_S d\sigma d\tau=\int_0^T d\tau\int_0^\pi d\sigma\{-\gamma[(X_\tau X_\sigma)^2-X_\tau^{\,2}X_\sigma^{\,2}]^{1/2}\}, \tag{2}$$

$$I_q=\int_0^T(L_1+L_2)d\tau, \tag{3}$$

$$L_i=\tfrac{1}{2}iX_{i\tau}^{\;\mu}(X_{i\tau}^{\;2})^{-1/2}\{\bar{\psi}_\tau\gamma_\mu\frac{\overleftrightarrow{\partial}}{\partial\tau}\psi_i\}-m_i(X_{i\tau}^{\;2})^{1/2}\bar{\psi}_i\psi_i, \tag{4}$$

and $X_\tau^{\;\mu}=\partial X^\mu/\partial\tau$, $X_\sigma^{\;\mu}=\partial X^\mu/\partial\sigma$, $X_1^{\;\mu}(\tau)=X^\mu(0,\tau)$, and $X_2^{\;\mu}(\tau)=X^\mu(\pi,\tau)$. The action (2) plus (3) governs the motion of the string (parametrized by $0<\sigma<\pi$) which keeps the quarks at $\sigma=0$ and $\sigma=\pi$. Angular brackets denote integration over the anticommuting field $\psi_i$ and $\bar{\psi}_i$ with no color component (see Berezin[3]). The integration over $X^\mu(\sigma,\tau)$ covers all the possible deformations of connected world sheets for a given initial and a final setup of the string. The gauge-fixing terms for $X^\mu(\sigma,\tau)$ and an initial and final conditions must be imposed in performing the integration. Then the formula defines a field theory of interacting string[4] if all possible topological sheets[5] are taken into account. It is interesting to point out that the quark Lagrangian (2) plus (3) is precisely the one previously proposed by Bars and Hanson,[6] without reference to the lattice gauge theory.

To prove the above statement, we begin by writing down the lattice-gauge action in Euclidean space:

$$I=\sum_{n,\pm\mu,\nu}g^{-2}\,\mathrm{Tr}(U_{n,\mu}U_{n+\hat{\mu},\nu}U_{n+\hat{\nu},\mu}^{\dagger}U_{n,\nu}^{\dagger}-1)+\sum_n[\tfrac{1}{2}K\sum_{\pm\mu}\bar{q}(n)(1+i\not{e}^\mu)U_{n,\mu}q(n+\hat{\mu})-\bar{q}(n)q(n)], \tag{5}$$

where $U$ and $q$ are the link and the colored-quark fields, respectively, and $e_\mu$ a unit vector which points in the $\mu$ direction.

The quantity we are concerned with is

$$P = \langle\langle \exp(I)\overline{D}(k_1, k_2)D(l_1, l_2)\rangle\rangle / \langle\langle \exp(I)\rangle\rangle, \tag{6}$$

where

$$D(l_1, l_2) = q(l_1)U_{l_1,\mu} \cdots U_{l_2-1,\mu}\overline{q}(l_2), \tag{7}$$

and the double-average notation denotes integration over both $q$ and $U$. According to Wilson,[1] we first choose a world sheet encircled by a perimeter, $l_1 \to l_2 \to k_2 \to k_1 \to l_1$, then perform the $U$ and $q$ integrations to obtain

$$P = \sum C^{1-n} \exp[-A \ln(g^2C)/a^2][\Pi_P(-K/2)(1+i\not{e}^\mu)], \tag{8}$$

where the summation extends over all world sheets, and $A$ represents the area of the world sheet, $\Pi_P$ the product along the perimeters $P$, $C$ the number of color components of the quark, and $n$ an integer which characterizes the topological structure of the world sheet, i.e., the number of holes plus twice that of the bundles of the sheet.

Now, we assume that the area $A$ can be replaced by the Nambu-Goto area factor[7] [Eq. (2)], with $\gamma \equiv \ln(g^2C)/\sqrt{2}a^2$ for $q$-$\overline{q}$ distances larger than $a$. Although the straightforward calculation provides a somewhat different area factor which violates the rotational invariance due to the rotational noninvariant representation of the original action (5), we assume that the invariance should be recovered in a refined formalism.

That the last factor in (8) can be replaced by the second factor in (1) can be confirmed if one notices that the following action reproduces the last factor in (8):

$$I_q = \sum_P [\sum_{\pm\mu} \tfrac{1}{2}K\overline{\psi}'(n)(1+i\not{e}_\mu)\psi'(n+\hat{\mu}) - \overline{\psi}'(n)\psi'(n)], \tag{9}$$

where the sum is taken over the lattice sites along $P$. Crucial differences between the second term in (5) and (9) are that $U$ is missing in (9) and that the latter contains series of $\psi'(n)$ only along the world line (the timelike direction in Lorentz space) while the former contains them along all directions. Since our model has local gauge invariance, we can always choose a special gauge where all links along the time direction are about unity. This enables us to take $\psi'(n+\hat{\mu}) \approx \psi'(n)$, if $\hat{\mu}$ points to a timelike direction, so that $\psi'(n+\hat{\mu})$ can be expanded about the site $n$, even in the strong-coupling approach. The expression (9) can then be replaced by the continuum limit:

$$\sum_P [\tfrac{1}{2}K\overline{\psi}' i\not{e}_\mu \overleftrightarrow{\partial^\mu}\psi' - (1-K)a^{-1}\overline{\psi}'\psi']\,dX^\mu = \int [\tfrac{1}{2}iX_\tau{}^\mu(\dot{X}_\tau{}^2)^{-1/2}\overline{\psi}\gamma_\mu \frac{\overleftrightarrow{\partial}}{\partial\tau}\psi - m\sqrt{X_\tau{}^2}\,\overline{\psi}\psi]\,d\tau, \tag{10}$$

where we have used $a = |dX^\mu|$, $e^\mu = X_\tau{}^\mu/\sqrt{X_\tau{}^2}$, $dX^\mu = X_\tau{}^\mu d\tau$, $\psi = \sqrt{K}\psi'$, and $m = (1-K)/aK$. We note that the continuum assumption (10) along the timelike direction has been necessary in obtaining the Hamiltonian formalism.[8] The above replacements show the equivalence between (8) and (1) if one changes the metric to Lorentz's.

At this point, we make some remarks. First, we have not yet succeeded in determining an exact functional measure in $X$. Second, the factor $C^{1-n}$ in (8) must be distributed over the wave-function renormalization constants by $\sqrt{C}$ and also over the coupling constants of string-string interactions by $1/\sqrt{C}$. In contrast to the usual string model where the coupling constant is arbitrary, the lattice gauge theory predicts the value. Third, we point out that the quark mass at the string ends, $m = (1-K)/aK$, is heavier than the one obtained by Wilson [who gives $(1-4K)/aK$]. The latter is considered to be the weak-coupling case. Finally, we note that the slope parameter $\gamma^{-1} = [\ln(Cg^2)/\sqrt{2}a^2]^{-1}$ does not tend to a finite value unless $g^2 \to 1$ as $a \to 0$. This seems to be inconsistent with asymptotic freedom. However, we disregard this feature under the assumption that the higher-order corrections by the interstring interactions will cure the behavior in $a \sim 0$, and proceed to examine the string properties.

Now, we solve the Lagrangian (2)–(4) by the classical method.[9] Euler's equations are

$$(\partial/\partial\tau)[\partial L_S/\partial X_\tau{}^\mu] + (\partial/\partial\sigma)[\partial L_S/\partial X_\sigma{}^\mu] = 0, \quad 0 < \sigma < \pi, \tag{11}$$

$$(\partial/\partial\tau)[\partial L_i/\partial X_{i\tau}{}^\mu] = \mp\, \partial L_S/\partial X_\sigma{}^\mu, \quad \sigma = 0, \pi, \tag{12}$$

for the string, and

$$(\partial/\partial\tau)[\partial L_i/\partial\bar{\psi}_{i\tau}] - \partial L_i/\partial\bar{\psi}_i = 0, \quad i=1,2, \tag{13}$$

for the quarks. In the timelike gauge $X^0(\sigma,\tau)=\tau$, the energy and the angular momentum turn out to be

$$H = \sum_i [m_i \bar{\psi}_i \psi_i (1-\vec{X}_{i\tau}^2)^{-1/2} - (1-\vec{X}_{i\tau}^2)^{-1}\vec{S}\cdot(\vec{X}_{i\tau\tau}\times\vec{X}_{i\tau})] + \gamma\int_0^\pi \{\vec{X}_\sigma^2/(1-\vec{X}_\tau^2)\}^{1/2} d\sigma, \tag{14}$$

$$\vec{J} = \sum_i \{m_i \bar{\psi}_i \psi_i (1-\vec{X}_{i\tau}^2)^{-1/2}(\vec{X}_i\times\vec{X}_{i\tau}) + \vec{S}_i - (1-\vec{X}_{i\tau}^2)^{-1}[(\vec{X}_i\cdot\vec{X}_{i\tau\tau})\vec{S}_i - (\vec{X}_i\cdot\vec{S}_i)\vec{X}_{i\tau\tau}]\} + \gamma\int_0^\pi \{\vec{X}_\sigma^2/(1-\vec{X}_\tau^2)\}^{1/2}(\vec{X}\times\vec{X}_\tau) d\sigma, \tag{15}$$

where $\vec{S}_i$ is the spin of the $i$th quark and is defined by[6]

$$\vec{S}_i = \tfrac{1}{4}(1-\vec{X}_{i\tau}^2)^{-1/2}\bar{\psi}_i\{\gamma_\mu X_{i\tau}{}^\mu, \vec{\sigma}\}\psi_i. \tag{16}$$

The Eqs. (13) for $\psi_i$ are homogeneous and can be solved for $\psi_i$ as a function of yet unknown coordinate $\vec{X}_i(\tau)$ and $\tau$. In the center-of-mass system of the string, a classical solution to (11), which is associated with the leading Regge trajectory, is known to be[9]

$$X^\mu(\sigma,\tau) = [\tau, \rho(\sigma)\cos\omega\tau, \rho(\sigma)\sin\omega\tau, 0], \tag{17}$$

$$\rho(\sigma) = \omega^{-1}\sin[\kappa\omega(\sigma-\sigma_0)]. \tag{18}$$

Using the solutions $\psi_i$ and (17), we determine the parameters $\kappa$ and $\sigma_0$, or equivalently $\rho_1=\rho(0)$ and $\rho_2=\rho(\pi)$, in such a way that (12) are fulfilled, namely,

$$[m_i(1-\omega^2\rho_i^2) \pm \tfrac{1}{2}\omega]\omega = (-1)^i\gamma[(1-\omega^2\rho_i^2)/(\omega\rho_i)]\,\mathrm{sgn}(\rho_i'), \tag{19}$$

where $\pm$ refers to the sign of $S_3$. The solution so obtained for a given spin configuration is substituted into (14) and (15), and the Regge trajectory is obtained by eliminating the angular frequency $\omega$. The intercepts with integer values of $J$ provide us with the hadron spectra. This method is equivalent to the Bohr-Sommerfeld quantization.

Although the full behavior of the trajectories has to be determined by numerical analysis (Fig. 1), some characteristic properties can be known by analytical method. As in the other string model,[9] the low- (high-) angular-momentum behavior is determined by large (small) $\omega$. The boundary condition (19) is then easily solved for these extreme cases. Some of the properties are listed below.

(1) Two natural-parity $[P=(-1)^J]$ and two unnatural-parity $[P=(-1)^{J+1}]$ trajectories are obtained.

(2) Two unnatural-parity ($\pi$ and $A_1$) trajectories are degenerate.

(3) The $\rho$-meson mass and the $\pi$-meson mass are degenerate and equal to the sum of constituent quark masses ($=m_1+m_2$) with no potential energy, i.e., the SU(6) symmetry.[10] Since the solutions of (19) for large $\omega$ are $\omega\rho_i \sim 2\gamma/\omega^2$, the trajectories behave in $H\sim m_1+m_2$ region as

$$J_\rho = 1 + (3/8\gamma^{2/3})(H-m_1-m_2)^{4/3}, \tag{20}$$

$$J_{\pi,A_1} = (1/2\pi\gamma)(H-m_1-m_2)^2, \tag{21}$$

$$J_{\epsilon'} = -1 + (1/4\pi\gamma)(H-m_1-m_2)^2. \tag{22}$$

(4) All trajectories become parallel and straight with the slope $(1/2\pi\gamma)$ in the high-mass limit.

(5) Effects of the quark mass and spin appear in the low-mass region. If the quark mass is large (as for the charmed quark), the slope around $J\sim 1$ becomes small. This is consistent with the $\psi$-family data.

As for baryons, we confirmed both the quark-mass additivity rule for ground states and the linear asymptotic behavior of the trajectories. These properties are considered to be a good 0th-order approximation to the hadron spectroscopy. We wish to emphasize that the spectrum has the SU(6) symmetry for ground states even though we used the relativistic approximation.

One has to keep in mind, however, that the low-mass region is not considered to be the best

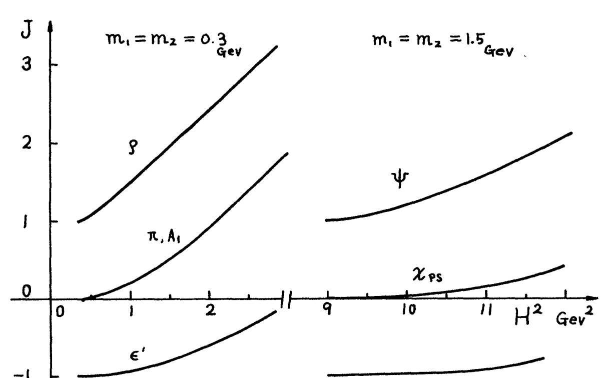


FIG. 1. Regge trajectories for mesons. Free parameters are quark masses and the asymptotic slope $(2\pi\gamma)^{-1}$ which is taken to be 1 GeV$^{-2}$.

place to test the lattice gauge model as long as one adopts the present approach, because (a) the string approximation may not be good when the length of the string ($\sim\gamma/\omega^3$) becomes of order of the lattice distance, and (b) the stationary-phase approximation that we used may not be valid in this case.

What is needed to get a better spectrum is suspected to be an additional spin-spin interaction. Although such an interaction cannot be obtained in the classical approximation of local string model, from the quantum corrections its appearance is highly expected in lattice gauge theories. If one supposes that the remnant interaction besides the string force is the weak-coupling exchange of the gluon between the quark and antiquark, one can confirm the existence of spin-spin interactions with the correct sign.

Detailed discussions for baryons and the quantum corrections will be published elsewhere.

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[1]K. G. Wilson, Phys. Rev. D 10, 2445 (1974); K. G. Wilson, in *Proceedings of the Conference on Gauge Theories and Modern Field Theory, Northeastern University, Boston, Mass., 1975,* edited by R. Arnowitt and P. Nath (MIT Press, Cambridge, Mass., 1975).

[2]J. Kogut, D. K. Sinclair, and L. Susskind, Cornell University Report No. CLNS-336, 1976 (to be published); T. Banks, S. Raby, L. Susskind, J. Kogut, D. R. T. Jones, P. N. Scharbach, and D. K. Sinclair, Cornell University Report No. CLNS-339, 1976 (to be published).

[3]F. A. Berezin, *The Method of Second Quantization* (Academic, New York, 1966).

[4]For the string theory without quarks, see, for example, S. Mandelstam, Nucl. Phys. B64, 205 (1973); M. Kaku and K. Kikkawa, Phys. Rev. D 10, 1110, 1823 (1974).

[5]Because of the existence of the antisymmetric tensor $\epsilon^{abc}$, there occurs a topologically different class of world sheets in the SU(3) color model, say, which are not included in those models in Ref. 4, i.e., the sheet swept out by strings with junctions. This class of strings are excluded for simplicity in the present article. Detailed discussions will be published elsewhere.

[6]I. Bars and A. J. Hanson, Phys. Rev. D 13, 1744 (1976); I. Bars, Phys. Rev. Lett. 36, 1521 (1976), and Nucl. Phys. B111, 413 (1976). Note, however, while Bars's $\psi$ carries color index, ours does not as a result of the $U$ integration [see below].

[7]Y. Nambu, unpublished; T. Goto, Prog. Theor. Phys. 46, 1560 (1971).

[8]J. Kogut and L. Susskind, Phys. Rev. D 11, 395 (1975).

[9]The method we use here has been applied to various string models by many authors. See, for example, A. Chodos and C. Thorn, Nucl. Phys. B72, 509 (1974); T. Takabayashi, in *Quantum Mechanics Determinism, Causality, and Particles,* edited by M. Flato (Reidel Publishing Co., Boston, Mass., 1976), p. 179; K. Kikkawa, M. Sato, and K. Uehara, Osaka University Report No. OU-GE 76-4, 1976 (to be published). (See also Ref. 6). $\psi$ is treated as a quantized field so that the spin remains unvanished in our approach.

[10]B. Sakita, Phys. Rev. 136, B1756 (1964); F. Gürsey and L. A. Radicati, Phys. Rev. Lett. 13, 173 (1964).