Some Exact Multipseudoparticle Solutions of Classical Yang-Mills Theory

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I present some exact solutions of the Polyakov-Belavin-Schwartz-Tyupkin equation $F_{\mu\nu} = \tilde{F}_{\mu\nu}$ for an SU(2) gauge theory in Euclidean space. My solutions describe a system with an arbitrary number of pseudoparticles, with arbitrary scale parameters and arbitrary separations, arranged along a line. The action for an n-pseudoparticle solution is precisely n times the action for a single pseudoparticle.

Recently Polyakov[1] made the remarkable suggestion that localized, finite-action solutions of the classical Euclidean equations of motion may dominate the Euclidean path integrals of quantum field theory. Belavin, Polyakov, Schwartz, and Tyupkin (BPST) described such a localized, finite-action solution for non-Abelian gauge theories[2]; it has become known as the pseudoparticle.

Here I will describe a much more extensive class of exact, analytic solutions for a classical SU(2) gauge theory in Euclidean space. My solutions have arbitrary integral values of the topological charge discovered by Belavin, Polyakov, Schwartz, and Tyupkin. They describe an assembly of the BPST pseudoparticles, with arbitrary scale parameters and arbitrary separations, but arranged along a line. My solutions may help clarify many-pseudoparticle effects which, as Polyakov suggested, may play an important role in the strong interactions.

Belavin, Polyakov, Schwartz, and Tyupkin showed that the fields of minimum action for fixed boundary conditions are solutions of $F_{\mu\nu} = \tilde{F}_{\mu\nu}$. If $F_{\mu\nu} = \tilde{F}_{\mu\nu}$, then in view of the Bianchi identity $D_\mu \tilde{F}_{\mu\nu} = 0$, the field equation $D_\mu F_{\mu\nu} = 0$ is also satisfied. My solutions will all satisfy $F_{\mu\nu} = \tilde{F}_{\mu\nu}$.

I will seek solutions of $F_{\mu\nu} = \tilde{F}_{\mu\nu}$ that are invariant under three-dimensional rotations combined with gauge transformations. I will call this a cylindrical symmetry, because it determines the dependence of the fields on the three-dimensional polar angles and leaves unknown only the dependence on the three-dimensional radius r and the Euclidean time t. The most general gauge field with cylindrical symmetry can be written as follows:

$$A_j^{\,a} = \frac{(\varphi_2 + 1)}{r^2}\epsilon_{jak}x_k + \frac{\varphi_1}{r^3}[\delta_{ja}r^2 - x_j x_a] + A_1\frac{x_j x_a}{r^2}, \qquad A_0^{\,a} = \frac{A_0 x^a}{r}. \tag{1}$$

Here, j and k refer to the three spatial dimensions, and a is the isospin index. The precise definitions of φ_1, φ_2, A_0, and A_1 are chosen for future convenience. These functions depend only on r and t. I will find the most general solution of $F_{\mu\nu} = \tilde{F}_{\mu\nu}$ that can be written in the form (1).

The *Ansatz* (1) is consistent with gauge transformations generated by a unitary matrix $U(x, t) = \exp[if(r, t)\vec{x}\cdot\vec{T}]$, where f is arbitrary and T_i are the generators of SU(2). This is an Abelian subgroup of the full gauge group. For the moment I avoid a choice of gauge. Given (1), one readily calculates the field tensor $F_{\mu\nu}^{\,a} = \partial_\mu A_\nu^{\,a} - \partial_\nu A_\mu^{\,a} - \epsilon^{abc}A_\mu^{\,b}A_\nu^{\,c}$:

$$F_{0i}^{\,a} = (\partial_0\varphi_2 - A_0\varphi_1)\frac{\epsilon_{iak}x_k}{r^2} + (\partial_0\varphi_1 + A_0\varphi_2)\frac{(\delta_{ai}r^2 - x_a x_i)}{r^3} + r^2(\partial_0 A_1 - \partial_1 A_0)\frac{x_a x_i}{r^4},$$
$$\tfrac{1}{2}\epsilon_{ijk}F_{jk}^{\,a} = -\frac{\epsilon_{ias}x_s}{r^2}(\partial_1\varphi_1 + A_1\varphi_2) + \frac{(\delta_{ai}r^2 - x_a x_i)}{r^3}(\partial_1\varphi_2 - A_1\varphi_1) + \frac{x_a x_i}{r^4}(1 - \varphi_1^2 - \varphi_2^2) \tag{2}$$

(where ∂_0 denotes $\partial/\partial t$ and ∂_1 denotes $\partial/\partial r$). The form of (2) suggests that I regard φ as a charged scalar interacting with the two-dimensional Abelian gauge field A_μ, with covariant derivative $D_\mu\varphi_i = \partial_\mu\varphi_i + \epsilon_{ij}A_\mu\varphi_j$. With integration over the polar angles, the action turns out to be

$$A = \tfrac{1}{4}\int d^3x\int dt\, F_{\mu\nu}^{\,a}F_{\mu\nu}^{\,a} = 8\pi\int_{-\infty}^{\infty}dt\int_0^\infty dr[\tfrac{1}{2}(D_\mu\varphi_i)^2 + \tfrac{1}{8}r^2F_{\mu\nu}^{\,2} + \tfrac{1}{4}r^{-2}(1 - \varphi_1^2 - \varphi_2^2)^2], \tag{3}$$

where $F_{\mu\nu}$ is of course $\partial_\mu A_\nu - \partial_\nu A_\mu$. This is very nearly the usual action for the two-dimensional

Abelian Higgs model; in fact, in curved space the action for the Abelian Higgs model is

$$\int d^2x\sqrt{g}\,[\tfrac{1}{2}g^{\mu\nu}D_\mu\varphi_i D_\nu\varphi_i+\tfrac{1}{8}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}+\tfrac{1}{4}(1-\varphi_1{}^2-\varphi_2{}^2)^2],$$

which agrees with (3) if $g^{\mu\nu}=r^2\delta^{\mu\nu}$. This metric corresponds to a space of constant negative curvature.

I now consider the equation $F_{\mu\nu}=\tilde{F}_{\mu\nu}$ or, equivalently, $F_{0i}{}^a=\frac{1}{2}\epsilon_{ijk}F_{jk}{}^a$. Equating corresponding terms in (2), I find

$$\partial_0\varphi_1+A_0\varphi_2=\partial_1\varphi_2-A_1\varphi_1,$$
$$\partial_1\varphi_1+A_1\varphi_2=-(\partial_0\varphi_2-A_0\varphi_1), \tag{4}$$
$$r^2(\partial_0A_1-\partial_1A_0)=1-\varphi_1{}^2-\varphi_2{}^2.$$

I will find the general solution of these equations. The key is the choice of gauge. I set $\partial_\mu A_\mu=0$, so that $A_\mu=\epsilon_{\mu\nu}\partial_\nu\psi$ for some ψ. The first two equations in (4) now become

$$[\partial_0-(\partial_0\psi)]\varphi_1=[\partial_1-(\partial_1\psi)]\varphi_2,$$
$$[\partial_1-(\partial_1\psi)]\varphi_1=-[\partial_0-(\partial_0\psi)]\varphi_2.$$

If one lets $\varphi_1=e^\psi\chi_1$, $\varphi_2=e^\psi\chi_2$, one finds simply

$$\partial_0\chi_1=\partial_1\chi_2,$$
$$\partial_1\chi_1=\partial_0\chi_2.$$

These are the Cauchy-Riemann equations, which say that $f=\chi_1-i\chi_2$ is an analytic function of $z=r+it$.

It remains to consider the third equation in (4). It becomes

$$-r^2\nabla^2\psi=1-f^*fe^{2\psi}. \tag{5}$$

Let us first note that this equation possesses a remaining gauge invariance. Consider the transformation

$$f\to fh,$$
$$\psi\to\psi-\tfrac{1}{2}\ln(h^*h), \tag{6}$$

where $h(z)$ is an analytic function. Because $\nabla^2\ln h^*h=0$ for any analytic function h (as long as h has no zeroes), (5) is invariant under this transformation. This invariance exists because the gauge condition $\partial_\mu A_\mu=0$ permits transformations $A_\mu\to A_\mu+\partial_\mu\lambda$, where $\nabla^2\lambda=0$. If h does have zeroes for $r>0$, then (6) introduces isolated singularities at those zeroes.

In order to solve (5), let $\psi=\ln r-\frac{1}{2}\ln(f^*f)+\rho$, where ρ is a new unknown function. By using the fact that $\nabla^2\ln f^*f=0$ for any analytic function f, except for isolated singularities that I momentarily ignore, (5) becomes simply

$$\nabla^2\rho=e^{2\rho}. \tag{7}$$

Equation (7) is called Liouville's equation.[3] Its general solution can easily be found by using conformal invariance. Let $\rho_1(z)$ be any particular solution of Liouville's equation; for example, $\rho_1(z)=-\ln[\frac{1}{2}(1-z^*z)]$. Now, consider an arbitrary analytic function $\omega(z)$. The Laplacian with respect to ω is $\nabla_\omega{}^2=|dz/dw|^2\nabla_z{}^2$ and ρ_1, as a function of ω, satisfies

$$\nabla_\omega{}^2\rho_1(\omega)=|dz/dw|^2e^{2\rho_1(\omega)}.$$

This is Liouville's equation (7) except for the factor $|dz/dw|^2$. Letting $\rho(\omega)=\rho_1(\omega)-\frac{1}{2}\ln|dz/dw|^2$, and using the fact that $\nabla^2\ln|dz/dw|^2=0$, I find that $\rho(\omega)$ satisfies the Liouville equation $\nabla_\omega{}^2\rho=e^{2\rho}$. Thus, if g is any analytic function, $\rho(z)=-\ln[\frac{1}{2}(1-g^*g)]+\frac{1}{2}\ln|dg/dz|^2$ satisfies Liouville's equation. This is, in fact, known to be the general solution of Liouville's equation.

Returning now to (5), the various singularities cancel if and only if $(dg/dz)/f$ has neither zeroes nor poles in the right half plane. This means that, up to a gauge transformation of type (6), the most general nonsingular solution of (5) is

$$\psi=-\ln\left(\frac{1-g^*g}{2r}\right),\quad f=\frac{dg}{dz}. \tag{8}$$

For ψ to be nonsingular, I must require $|g|=1$ for $r=0$ and $|g|<1$ for $r>0$. The most general analytic function with these properties and smooth behavior for $z\to\infty$ is

$$g(z)=\prod_{i=1}^{k}\left(\frac{a_i-z}{a_i{}^*+z}\right), \tag{9}$$

where the a_i are an arbitrary set of complex numbers (some perhaps equal) constrained only to have $\mathrm{Re}\,a_i>0$. (8) and (9) provide the most general solution of $F_{\mu\nu}=\tilde{F}_{\mu\nu}$ with cylindrical symmetry and finite action.

Let us now consider the physical content of these solutions. In view of the gauge invariance (6), the only gauge-invariant property of f is the location of its zeroes in the right half plane. The zeroes of f in the right half plane may therefore play a central role.

If $k=1$ in (9), f has no zeroes and an easy calculation shows that this solution is a gauge transform of the vacuum. If $k=2$, f has precisely one zero in the right half plane. This field describes the BPST pseudoparticle (but in a different gauge). The imaginary part of the zero of f determines the location of the pseudoparticle along the time axis, while the real part determines the pseudoparticle scale.

For general k, the total multiplicity of the zeroes of f in the right half plane is always $k-1$. The natural generalization of the comments in the last paragraph would be that for general k, my solution describes $k-1$ pseudoparticles, with real and imaginary parts determined by the zeroes of f.

I will verify this, but first there is a counting problem to consider. For $k=2$ my solution (8) and (9) involves four real parameters—the real and imaginary parts of a_1 and a_2. But the BPST pseudoparticle, with cylindrical symmetry, has only two parameters—the position along the time axis and the scale. For general k my solution has $2k$ real parameters, but I expect the physics to involve only $k-1$ positions and $k-1$ scales. The explanation is that my solution (8) and (9) still possesses a remaining two-parameter gauge invariance. If one replaces f and g by

$$\tilde{g}=\frac{c+g}{c^*g+1}, \quad \tilde{f}=\frac{d\tilde{g}}{dz}, \tag{10}$$

where $|c|<1$, then $\tilde{g}$ is still of the form (9), and the transformation from f and g to $\tilde{f}$ and $\tilde{g}$ is a gauge transformation of type (6). Also, $\tilde{f}$ and f have the same zeroes. Consequently, the physics in (8) and (9) depends not on the k complex number a_i but only on $k-1$ complex functions of them —the zeroes of f.

I now wish to verify two facts: That the solution (8) and (9) always has a topological charge equal to $k-1$, and that, as the zeroes of f become widely separated, my solution describes $k-1$ widely spearated, BPST pseudoparticles.

I first consider what happens as the zeroes of f become widely separated. It is essential to choose the right gauge. Let us keep fixed one of the zeroes of f, α_0, which I assume to be a simple zero, and let the distance from α_0 to the other zeroes, $\alpha_1, \dots, \alpha_m$, become large. In a general gauge, there is no simple relation between the zeroes of g and the zeroes of f, but by a gauge transformation of type (10), I may always arrange it so that α_0 is one of the zeroes of g. It will be a double zero, since $f=dg/dz$ has a simple zero at α_0. So g has the form

$$g=[(\alpha_0-z)/(\alpha_0^*+z)]^2\prod_{i=1}^{m}[(\beta_i-z)/(\beta_i^*+z)].$$

The β_i are complicated functions of the α_i. But as the differences $|\alpha_i-\alpha_0|$ become large, the differences $|\beta_i-\alpha_0|$ also become large. Then for z in the neighborhood of α_0, the entire factor $\pi[(\beta_i-z)/(\beta_i^*+z)]$ may be set equal to the constant $\pi(\beta_i/\beta_i^*)$. Dropping this phase factor I find that as the $|\alpha_i-\alpha_0|$ become large, with z fixed, I may approximate g by $[(\alpha_0-z)/(\alpha_0^*+z)]$. But this is the special case of (9) for $k=2$, which is already known to describe a single BPST pseudoparticle with scale and location controlled by α_0. Thus, in this limit, my solution describes a system of isolated BPST pseudoparticles, one for each zero of f. Hence, the topological charge $(\frac{1}{8}\pi^2)\int d^4x F_{\mu\nu}\tilde{F}_{\mu\nu}$ equals the number of zeroes of f, at least if those zeroes are widely separated. Since a solution with nearby zeroes of f can be reached continuously from a solution with widely separated zeroes, the topological charge equals the number of zeroes whether they are separated or not.

It is instructive to derive this result in a more direct way. From expression (2) for $F_{\mu\nu}$, I find that the four-dimensional topological charge $(\frac{1}{8}\pi^2)\int d^4x F_{\mu\nu}\tilde{F}_{\mu\nu}$ becomes

$$(1/2\pi)\int d^2x[\epsilon_{\mu\nu}\epsilon_{ij}D_\mu\varphi_i D_\nu\varphi_j+\tfrac{1}{2}\epsilon_{\mu\nu}F_{\mu\nu}(1-\varphi^2)].$$

By algebraic manipulations this can be rewritten

$$(1/2\pi)\int d^2x[\partial_\mu(\epsilon_{ij}\epsilon_{\mu\nu}\varphi_i D_\nu\varphi_j)+\tfrac{1}{2}\epsilon_{\mu\nu}F_{\mu\nu}].$$

The first term is a total divergence, and so can be written as a boundary integral which vanishes because for my solutions $D_\mu\varphi_i=0$ at the boundary of the space. The second term is also a total divergence, but its integral does not necessarily vanish—it is the usual topological charge of the Abelian Higgs model. Thus for cylindrically symmetric fields, I can identify the four-dimensional topological charge discovered by Belavin, Polyakov, Schwartz, and Tyupkin with the two-dimensional topological charge of the Higgs model.

For the Higgs model, $(1/4\pi)\int d^2x\epsilon_{\mu\nu}F_{\mu\nu}$ equal $1/2\pi$ times the change in phase of $\varphi=\varphi_1-i\varphi_2$ around a contour that encloses the region in which the fields differ significantly from the vacuum. With $\varphi=fe^\psi$, this is

$$\frac{1}{2\pi i}\oint ds\frac{d}{ds}\ln(fe^\psi)=\frac{1}{2\pi i}\oint ds\frac{d}{ds}\ln f+\frac{1}{2\pi i}\oint ds\frac{d\psi}{ds},$$

where ds is the line element along the contour. The second term vanishes identically (since ψ, unlike $\ln f$, is a single-valued function); and the first term, by the argument principle of complex variable theory, is equal to the number of zeroes of f within the contour. This shows that the charge equals the number of zeroes of f.

A final remark is in order. My solutions possess finite action and finite $F_{\mu\nu}{}^a$, but in the gauge in which I am working, the four-dimensional gauge field $A_\mu{}^a$ is actually singular at $r=0$. This is obvious from (1), where I see that the field is nonsingular only if $\varphi_2=1$ and $\varphi_1=0$ at $r=0$. It is necessary to perform a gauge transformation on the solutions to satisfy these conditions; such a transformation always exists because I have $\varphi^2=1$ at $r=0$. In the language of (6), (8), and (9), a suitable gauge function is

$$h=-i\prod_{i=1}^{k}(a_i{}^*+z)^2.$$

Thus,

$$\psi=\ln\frac{2r}{(1-g^*g)(h^*h)^{1/2}}$$

and

$$\varphi_1-i\varphi_2=h\frac{dg}{dz}e^{\psi}$$

give nonsingular four-dimensional gauge fields that satisfy the equations of motion.

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Where is the Dip Structure in pp Elastic Scattering?

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The structureless $\exp(1.8t)$ behavior of recent pp data for $|t|\gtrsim 2$ GeV^2, $\sqrt{s}=53$ GeV, is shown to be in sharp disagreement with dip-structure predictions from popular models of diffraction. Flip amplitudes, real parts, and large-angle effects are quantitatively insufficient to resolve the discrepancy. Modifications of some familiar ideas on diffraction (like eikonalization) seem necessary.

Recently, the CERN-Hamburg-Orsay-Vienna (CHOV) collaboration published accurate results on pp elastic scattering at $\sqrt{s}=53$ GeV extending out to $|t|\simeq 9$ GeV^2.[1] This is a substantial extension of previous results, which were limited to $|t|\lesssim 3$ GeV^2.[2] The purpose of this Letter is to show how the t dependence of the new data necessitates modification of current ideas on diffraction scattering.

The CHOV data have two noteworthy features (see Fig. 1): (i) $d\sigma/dt$ has the well-known dip at $|t|\simeq 1.3$ GeV^2, but there is *no additional second dip* below $|t|\simeq 7$ GeV^2. (ii) The data are essentially *a structureless exponential* beyond the first maximum (at $|t|\simeq 2$ GeV^2) with a slope $B_2=1.8$ GeV^{-2}—considerably smaller than a typical slope $B_1\simeq 12$ GeV^{-2} in the forward peak.

The above features are in sharp conflict with the expectations of currently popular models[3] of high-energy diffraction scattering (which I shall also refer to as the Pomeron). I now demonstrate this disagreement by looking at various models which have some physical basis and more or less agree with previous ($|t|\lesssim 3$ GeV^2) data.

The pp elastic amplitude is customarily given by[3]

$$A(s,t)=P(s,t)+C(s,t). \qquad (1)$$

The Pomeron contribution $P(s,t)$ is approximately pure imaginary, dominates at small angles, and contains dip structure. The large-angle contribution $C(s,t)$ is smoothly behaved (no dips) and approximately real for pp scattering. The real phase is established by use of dispersion relations[4] or derivative analyticity relations.[5] It can be understood in the duality framework, since the pp channel is exotic. Since P and C are approximately out of phase, no significant interference occurs and $d\sigma/dt\propto P^2+C^2$.

Most current Pomeron models have a single, pure imaginary amplitude and can be roughly classified into two categories.

(A) Models with an s-channel viewpoint.—Such models are usually described by the amplitude