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Exact Linearization of a Painlevé Transcendent

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There is a connection between nonlinear partial differential equations that can be solved by the inverse scattering transform and nonlinear ordinary differential equations without movable critical points (e.g., Painlevé transcendents). We exploit this connection to reduce the second equation of Painlevé to a linear integral equation. We also describe a class of nonlinear ordinary differential equations that can be exactly linearized by this method.

Fundamental work was done at the turn of the century by Painlevé and Gambier, who studied ordinary differential equations of the form (A)

$$\frac{d^2w}{dz^2} = F(z, w, dw/dz),$$

where F is analytic in z, algebraic in w, and rational in dw/dz. (A thorough study of this work is given by Ince.[1]) They identified *all* equations of the form (A) for which the solutions have no movable critical points [i.e., the locations of any branch points or essential singularities do not depend on the constants of integration of (A)]. This classification includes all suitable linear equations, equations for elliptic functions, and six equations which define new transcendental functions, known as the six *Painlevé transcendents*. The first three of these are

$$d^2w/dz^2 = 6w^2 + z; \tag{1}$$

$$d^2w/dz^2 = 2w^3 + zw + \alpha; \tag{2}$$

$$\frac{d^2w}{dz^2} = \frac{1}{w}\left(\frac{dw}{dz}\right)^2 - \frac{1}{z}\frac{dw}{dz} + \frac{1}{z}(\alpha w^2 + \beta) + \gamma w^3 + \frac{\delta}{w}. \tag{3}$$

These equations are said to be *irreducible*, because they cannot be reduced to simpler (ordinary differential) equations or combinations thereof. This fact made the main result of this Letter [that (2) reduces to a linear integral equation] that much more surprising to us.

We demonstrate a close connection between these nonlinear ordinary differential equations without movable critical points and nonlinear partial differential equations that can be linearized exactly by an inverse scattering transform (IST). The Boussinesq equation,

$$u_{tt} - u_{xx} - 6(u^2)_{xx} + u_{xxxx} = 0, \tag{4}$$

was linearized exactly with IST.[2] We observe that (4) has a self-similar solution of the form $u(x, t) = w(x - t)$. Depending on the choice of constants of integration, either $w(z)$ is an elliptic function or it satisfies (1). In either case, it has no movable critical points.

The modified Korteweg-de Vries equation,

$$u_t - 6u^2u_x + u_{xxx} = 0, \tag{5}$$

was linearized exactly with IST.[3] It has a similarity solution of the form $u(x,t)=(3t)^{-1/3}w(x\times(3t)^{-1/3})$; $w(z)$ satisfies (2), and has no movable critical points. Similarly, the Korteweg-de Vries equation,

$$u_t+6uu_x+u_{xxx}=0, \tag{6}$$

was linearized exactly with IST,[4] and can be transformed into (5).[5] Consequently, its similarity solution can be transformed into the solution of (2).[6] Moreover, the Bäcklund transformation between (5) and (6) is a Ricatti equation, and therefore has no movable critical points.[1]

The sine-Gordon equation,

$$u_{xt}=\sin u, \tag{7}$$

was linearized exactly with IST.[7] It has a similarity solution[8] of the form $u(x,t)=f(xt)$. Under the transformation, $w=\exp(if)$, $w(z)$ satisfies (3) for an appropriate choice of constants, and has no movable critical points.

The connection between these special partial differential equations and the equally special ordinary differential equations appears to be basic. It provides information about *both* the ordinary and the partial differential equations which is not readily available by other means. The primary purpose of this paper is to exploit this connection to analyze the second Painlevé transcendent, but the methods are not restricted to this problem.

We consider a special case of (2):

$$d^2w/dz^2=2w^3+zw. \tag{8}$$

In the limit, $w\to 0$, this equation reduces to the Airy equation and its solution may be thought of as a nonlinear generalization of an Airy function [$\mathrm{Ai}(z)$]. In this regard, it provides the archetype of a "nonlinear turning point," just as the Airy equation does in linear problems. Consequently, it is important to determine whether (8) has any solutions that are bounded for all real z, and if so, to "connect" the behavior as $z\to-\infty$ to that as $z\to+\infty$. This global information can be found rather directly, after embedding the problem within the appropriate partial differential equation.

There is a one-parameter family of real solution of (8) that are bounded for all real z. The asymptotic behavior of these solutions can be found by a local analysis to be

$$w(z)\sim r\mathrm{Ai}(z)\sim\frac{r}{2\sqrt{\pi}}z^{-1/4}\exp(-\tfrac{2}{3}z^{3/2}),\quad z\to+\infty;$$
$$w(z)\sim d|z|^{-1/4}\sin(\tfrac{2}{3}|z|^{3/2}-\tfrac{3}{4}d^2\ln|z|+\theta), \tag{9}$$
$$z\to-\infty,$$

where r, d, and θ are constants. The connection problem is to find $d(r)$ and $\theta(r)$. By analyzing the long-time behavior of the solution of (6), we found[9] that for $-1<r<1$

$$d^2=-\pi^{-1}\ln(1-r^2). \tag{10}$$

For $|r|\geq 1$, the solution of (8) is not bounded for all real z. $\theta=\theta(r)$ was not determined by the method used by Ablowitz and Segur.[9]

The main result in this paper is that the connection between (5) and (8) not only provides practical formulas, such as (10), but actually reduces (8) to the following set of *linear* integral equations (for $y>x$, r real):

$$K_1(x,y)-r\mathrm{Ai}\left(\frac{x+y}{2}\right)+\frac{r}{2}\int_x^\infty K_2(x,s)\mathrm{Ai}\left(\frac{s+y}{2}\right)ds=0,\quad K_2(x,y)+\frac{r}{2}\int_x^\infty K_1(x,z)\mathrm{Ai}\left(\frac{z+y}{2}\right)dz=0, \tag{11}$$

where $w(x)=K_1(x,x)$. If desired, $K_2(x,y)$ can be eliminated from (11) to yield a single equation for $K_1(x,y)$. These equations were obtained by seeking a purely self-similar solution to the linear integral equations that arise in the inverse scattering solution of (5). In the present context, we consider them as equations in their own right. The first step is to establish the validity of (11), at least in a region of space.

Claim.—Let $|r|<1$ and $x\geq\frac{1}{9}$. Then there is a unique solution of (11) that is square-integrable on $[x,\infty)$. This solution is a continuous function of y.

Proof.—(i) For $z\geq\frac{1}{9}$,

$$0<\mathrm{Ai}(z)<\tfrac{1}{3}\exp(-\tfrac{2}{9}z). \tag{12}$$

Consequently, one can show that {in L_2-norm on $[x,\infty)$},

$$\left\|\tfrac{1}{2}\int_x^\infty\varphi(z)\mathrm{Ai}\left(\frac{z+y}{2}\right)dz\right\|^2\leq\|\varphi\|^2.$$

Uniqueness follows from using this result to show that the homogeneous version of (11) has only the trivial solution.

(ii) There is a formal Neumann series solution to (11). For $K_1(x, y)$,

$$v_0(x, y) = -r\mathrm{Ai}\left(\frac{x+y}{2}\right), \quad v_{n+1}(x, y) = \frac{r^2}{4}\int\int_x^\infty v_n(x, z_1)\mathrm{Ai}\left(\frac{z_1+z_2}{2}\right)\mathrm{Ai}\left(\frac{z_2+y}{2}\right)dz_1\,dz_2,$$
$$K_1(x, y) = \sum_0^\infty v_n(x, y). \tag{13}$$

Using (12), one shows that $\|v_{n+1}\| < r^2\|v_n\|$, so that the series converges for $0 \leq r^2 < 1$, and defines the solution of (11). Moreover, the convergence is uniform for $y \in [x, \infty)$.

(iii) Continuity of the solution follows from subtracting the solution of (11) at (x, y) from that at $(x, y + \Delta)$, and utilizing the smooth, monotonic behavior of $\mathrm{Ai}(z)$. This completes the proof.

The same proof applies for any real r, if x is large enough. Based on the result in Ref. 9, we conjecture that if $|r| < 1$, (11) has a unique, continuous solution for any finite real x. Moreover, we expect that if $|r| > 1$ a pole exists at some finite real x, and for $r = 1$, the critical branch is obtained. In any case, we now state the main result.

Theorem.—There is an x_0 such that for $x_0 \leq x \leq y$, $K_1(x, y)$, defined by (11), satisfies

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 K_1(x, y) = \left(\frac{x+y}{2}\right)K_1(x, y) + 2[K_1(x, x)]^2 K_1(x, y). \tag{14}$$

In particular, $K_1(x, x)$ satisfies (8), so that this second Painlevé transcendent is also the solution of a well-behaved, linear integral equation.

Proof.—(i) Define the linear operator

$$L = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 - \frac{x+y}{2}. \tag{15}$$

By direct computation, using (13), the identity $x + y = (x + z_1) - (z_1 + z_2) + (z_2 + y)$, and repeated integration by parts, one shows that

$$Lv_0 = 0, \quad Lv_{n+1}(x, y) = \frac{r^2}{4}\int\int_x^\infty [Lv_n(x, z_1)]\mathrm{Ai}\left(\frac{z_1+z_2}{2}\right)\mathrm{Ai}\left(\frac{z_2+y}{2}\right)dz_1\,dz_2$$
$$-r^2\mathrm{Ai}\left(\frac{x+y}{2}\right)\frac{d}{dx}\int_x^\infty v_n(x, z)\mathrm{Ai}\left(\frac{z+x}{2}\right)dz, \tag{16}$$

$$r\frac{d}{dx}\int_x^\infty v_n(x, z)\mathrm{Ai}\left(\frac{z+x}{2}\right)dz = -2\sum_0^n v_j(x, x)v_{n-j}(x, x). \tag{17}$$

(ii) Using these results in an induction argument establishes, for $n \geq 0$,

$$Lv_0(x, y) = 0, \quad Lv_{n+1}(x, y) = 2\sum_{j=0}^{n}\sum_{k=0}^{n-j} v_j(x, x)v_k(x, x)v_{n-j-k}(x, y). \tag{18}$$

(iii) Define

$$W_N(x, y) = \sum_0^N v_n(x, y). \tag{19}$$

Summing the results in (18) yields

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 W_N(x, y) = \left(\frac{x+y}{2}\right)W_N(x, y) + 2\sum_n^{N-1}\sum_j^n\sum_k^{n-j} v_j(x, x)v_k(x, x)v_{n-j-k}(x, y). \tag{20}$$

On the right-hand side, the limit $(N \to \infty)$ exists. On the left-hand side, termwise differentiation of the series is justified by the uniform convergence of W_n and its derivatives. This proves (14). Reduction to (8) is accomplished by taking $y = x$, and this completes the proof.

These ideas can be extended to a more general class of nonlinear ordinary differential equations. The simplest estension is to change the sign of $2w^3$ in (8). The procedure is unchanged, but changes certain signs in (10) and in (11).

More generally, we define the kernels (for $n \geq 1$)

$$A_n(x) = 2\pi^{-1}\int_{-\infty}^{\infty}\exp[ikx + ik^{2n+1}/(2n+1)]dk. \quad (21)$$

For $n=1$, $A_1(x) = \mathrm{Ai}(x)$. Replacing $\mathrm{Ai}(x)$ in (11) by $A_n(x)$ allows us to linearize

$$(M)^n(dw/dz) + d(zw)/dz = 0, \quad (22)$$

where

$$M = -\frac{d^2}{dz^2} - 4\frac{dw}{dz}\int_z^{\infty} dy\, w(y) + 4w^2,$$

and $w \to 0$ as $z \to \infty$. These are the similarity equations $[x(t/a)^{-a} \to x,\ a = (2n+1)^{-1}]$ associated with the evolution equations of Ablowitz *et al.*[10] with $\omega(k) = -k^{2n+1}$, $r = q$.[10]

Presumably, formulas such as (10) now can be established directly from (11). From our viewpoint, the most important consequence of these results is that they establish a connection between nonlinear partial differential equations that can be solved by IST and nonlinear ordinary differential equations without movable critical points. The precise nature and range of this connection will be the subject of future research.

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Supergravity and Square Roots of Constraints*

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The algebra of the weakly vanishing Hamiltonian generators of localized translations, rotations, and supersymmetry transformations of supergravity theory is found. I argue that supergravity is the square root of ordinary general relativity in the same way as the Dirac equation is the square root of the Klein-Gordon equation and the Ramond-Neveu-Schwarz model is the square root of the spinless string.

There is a close relationship between taking the square root of the constraints of a Hamiltonian system, introducing spin degrees of freedom in a natural manner into a physical theory, and the idea of supersymmetry.[1] Here I call supersymmetry the invariance of a theory under a transformation which mixes Fermi variables (obeying anticommutation rules) with Bose variables (obeying commutation rules). The simplest example of this relationship is the Dirac electron, which regarded as a constrained Hamiltonian system possesses two constraints which may be taken to be

$$\theta_\mu p^\mu + \theta_5 m \approx 0 \quad (1a)$$

and

$$\mathcal{H} = p_\mu p^\mu + m^2 \approx 0, \quad (1b)$$

where $\theta_\mu = i\gamma_5\gamma_\mu/\sqrt{2}$ and $\theta_5 = \gamma_5/\sqrt{2}$. These constraints are closed, in the sense

$$\{\mathcal{S}, \mathcal{S}\} = -\mathcal{H}, \quad (2a)$$

$$[\mathcal{S}, \mathcal{H}] = 0, \quad (2b)$$

$$[\mathcal{H}, \mathcal{H}] = 0, \quad (2c)$$

and the quantum theory is obtained by demanding that physical states must be annihilated by $\mathcal{S}$ and $\mathcal{H}$. On account of (2a), I have $\mathcal{S}|\psi\rangle = 0 \Rightarrow \mathcal{H}|\psi\rangle = 0$ and I say that the Dirac equation is the square