perimental data deviate strongly from this trend in the 100–200-MeV region. Above 250 MeV the measured cross sections are seen to exceed the single-step predictions by several orders of magnitude. This discrepancy is far beyond the range of theoretical uncertainty in the single-step cross section ensuing from reasonable variations in the potential well parameters. It provides strong evidence for the involvement of more than one nucleon in the photon absorption mechanism and, hence, the possibility of discovering the details of the interaction processes which provide the necessary additional high-momentum components.

Two such processes have already been investigated theoretically in a qualitative way and are shown to be capable of enhancing the (γ,p) cross section above 100 MeV, viz. short-range correlations[6] due to the repulsive core of the internucleon force and a two-step mechanism[5] in which the $\Delta(1232)$ nucleon isobar is excited in an intermediate state [see Fig. 2(b)]. The preliminary results of this latter calculation are shown in Fig. 1. It is evident that the Δ excitation mechanism can make a major contribution in the 100–300-MeV photon energy region.

Experimental data of reasonable accuracy and extent are now available over the kinematic range in which one might hope to observe short-range effects in the (γ,p) reaction. Because of the apparent importance of virtual Δ excitation, however, a more careful theoretical treatment of this and other processes[3,5,6] is necessary before additional constraints on the internucleon force at small distances may be obtained.

We wish to thank Dr. Londergan for sending us his calculations prior to publication.

*Supported in part by U. S. Energy Research and Development Administration under Contract No. E(11-1)-3069.

†Supported in part by the Science Research Council.

[1]J. L. Mathews, D. J. S. Findlay, S. N. Gardiner, and R. O. Owens, Nucl. Phys. A267, 51 (1976), and earlier references quoted therein.

[2]D. J. S. Findlay and R. O. Owens, to be published.

[3]H. Hebach, A. Wortberg, and M. Gari, Nucl. Phys. A267, 425 (1976).

[4]The photoprotons from beryllium are kinematically excluded from this part of the spectrum.

[5]G. D. Nixon, J. T. Londergan, and G. E. Walker, Bull. Am. Phys. Soc. 21, 67 (1976), and to be published.

[6]A. Malecki and P. Picchi, Lett. Nuovo Cimento 8, 16 (1973).

Nuclear Sp(3,*R*) Model

G. Rosensteel
Department of Physics, McMaster University, Hamilton, Ontario, Canada L8S4M1

and

D. J. Rowe
Department of Physics, University of Toronto, Toronto, Ontario, Canada M5S1A7
(Received 13 May 1976)

A microscopic model is presented which provides a practical means for selecting the states necessary for the development of nuclear collective rotational and quadrupole vibrational motions in a shell-model calculation. The model is based on the noncompact Sp(3,*R*) algebra and is a natural generalization of Elliott's SU(3) model to include many major shells.

In spite of the enormous successes of the nuclear rotational model, a microscopic theory of rotational states has proved extraordinarily elusive. One of the problems is to learn how to recognize rotational states. In a recent paper[1] we proposed a criterion for designating a state rotational based on the concept of a *well-defined intrinsic shape,* measurable with *shape observables.*

The essential idea follows a suggestion of Baranger.[2] One observes that each set of nucleon coordinates defines a traceless quadrupole mass tensor Q and hence a set of principal axes and principal values. Thus the nuclear density $|\psi(\vec{r}_1, \ldots, \vec{r}_A)|^2$ defines a probability distribution $P(\lambda_1, \lambda_2, \lambda_3)$ for the principal values of the quadrupole mass tensor. The criterion for a state to be rotational is then that the width of the distribution in λ_k should be small compared to its mean value It was shown that $\hat{\lambda}_k$ can be expressed as a function $\hat{\lambda}_k(\hat{a}_2, \hat{a}_3)$ of two scalar operators

$$\hat{a}_2 \equiv -\tfrac{1}{2}\mathrm{Tr}(\hat{Q}^2) = -\tfrac{1}{12}(\hat{Q}\times\hat{Q})^0, \qquad (1)$$

$$\hat{a}_3 \equiv -\det(\hat{Q}) = \tfrac{1}{108}(\tfrac{7}{2})^{1/2}(\hat{Q}\times\hat{Q}\times\hat{Q})^0, \qquad (2)$$

where cross products signify angular momentum coupling and carets denote operators. Since an eigenstate of $\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3$ is necessarily also an eigenstate of $\hat{a}_2$ and $\hat{a}_3$ the criterion for a state to have a well-defined intrinsic shape can be expressed by the inequalities

$$\langle \hat{a}_2{}^2\rangle - \langle \hat{a}_2\rangle^2 \ll \langle \hat{a}_2\rangle^2, \quad \langle \hat{a}_3{}^2\rangle - \langle \hat{a}_3\rangle^2 \ll \langle \hat{a}_3\rangle^2. \qquad (3)$$

It is shown that this criterion is indeed satisfied by the adiabatic rotational model.

The objective of this Letter is to show that this criterion also provides the means to generate microscopic rotational wave functions in a shell-model basis. All we have to do is find simultaneous eigenstates of the commuting operators $\hat{a}_2$, $\hat{a}_3$ and the angular momentum operators $\hat{L}^2$ and $\hat{L}_z$.

Now $\hat{a}_2$ and $\hat{a}_3$ are the Casimir invariants of the algebra $[R^5]$SO(3), whose generators are $\vec{\mathrm{L}}$ and $\hat{Q}$. Thus the irreducible representations of $[R^5]$SO(3) are pure rotational states according to the definition (3). In fact the irreducible representations of $[R^5]$SO(3) have been determined by Ui[3] and Weaver, Biedenharn, and Cusson[4] and shown to reproduce the rotational-model predictions for $E2$ transitions. Unfortunately, the problem of realizing irreducible representations of $[R^5]$SO(3) on shell-model state space is nontrivial due to the fact that exact eigenstates of $\hat{a}_2$ and $\hat{a}_3$ are non-normalizable (i.e., they are δ functions in λ) and cannot be expanded in a finite shell-model basis. However, the criterion (2) does not demand *exact* eigenstates and indeed, on physical grounds, we expect some vibrational shape fluctuations. It is reasonable therefore, to seek eigenstates of $\hat{a}_2$, $\hat{a}_3$, $\hat{L}^2$, and $\hat{L}_z$ in a truncated shell-model space of say r harmonic-oscillator shells. Furthermore, the admixtures of high-lying shell-model configurations in low-lying physical states should be small if the shell-model makes any kind of sense.

The rather drastic truncation of keeping only states from a single $(0\hbar\omega)$ harmonic oscillator (HO) shell results in states that belong to a single irreducible representation of SU(3). This is clear since, when truncated to a single shell, the operators $\hat{a}_2$, $\hat{a}_3$, $\hat{L}^2$, and $\hat{L}_z$ are in the enveloping algebra of SU(3). However, these states differ somewhat, in general, from Elliott's[5] SU(3) basis states. They are in fact identical to the SU(3) basis states of Bargmann and Moshinsky[6] and Judd *et al.*,[7] which, according to our criterion, are the closest possible approximations to rotational states that exist within a single shell.

If we attempt to enlarge the space to include the $(0\hbar\omega, 2\hbar\omega, \ldots, 2r\hbar\omega)$ HO shells, the shell-model dimensions rapidly become astronomical and the only hope for progress is to find another algebraic structure [like SU(3) for $r=0$] to limit the dimensions.

Considerations of nuclear quadrupole dynamics suggest that $[R^6]\mathrm{SL}(3,R)$, or CM(3) as it has been named,[8,9] may be a suitable algebra. The generators of CM(3) are $\vec{\mathrm{L}}$, $\hat{Q}$, and $\hat{S}$, where $\hat{Q}$ may here have nonzero trace and

$$\hat{S} = -(i/h)[\hat{Q}, H] \qquad (4)$$

is the shear momentum tensor, whose components are the generators of linear incompressible deformations. Thus CM(3) involves the vibrational degrees of freedom in addition to the rotational degrees of freedom of its subalgebra $[R^5]$SO(3). Actually, CM(3) is but an algebraic statement of the Bohr model generalized to allow large-amplitude vibrations. Furthermore, it has been shown[10] that, with some reasonable assumptions about the two-nucleon interaction, a canonical transformation to collective and intrinsic coordinates can be made such that the Hamiltonian separates cleanly into collective and intrinsic parts with H_{coll} a rational function of the generators of CM(3).

The irreducible representations of CM(3) have been determined.[9] But again it appears that the decomposition of shell-model states into irreducible CM(3) subspaces is too difficult to carry out at this time.

The appropriate generalization is the symplectic Lie algebra $\mathrm{Sp}(3,R)$ which is the smallest algebra containing both $[R^5]$SO(3) and SU(3). Fortunately, $\mathrm{Sp}(3,R)$ also includes CM(3) as a subalgebra so that the Bohr model is also realized,

$$\mathrm{Sp}(3,R) > \mathrm{CM}(3) > [R^5]\mathrm{SO}(3), \quad \mathrm{Sp}(3,R) > \mathrm{SU}(3).$$

This subalgebra structure is vital to both the physical significance of the symplectic model and its ultimate practicality. On the one hand, the important collective degrees of freedom treated phenomenologically by the Bohr and collective rotational models are incorporated in $\mathrm{Sp}(3,R)$

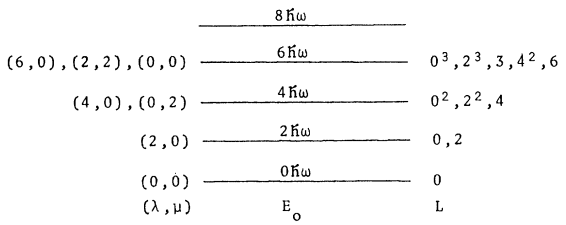

FIG. 1. SU(3) irreducible representations (λ,μ) spanning the Sp$(3,R)$ irreducible representation whose $0\hbar\omega$ subspace transforms as $(0,0)$.

through the CM(3) and $[R^5]$SO(3) subalgebras. On the other hand, the Elliott SU(3) subalgebra takes care of the shell-model aspect of the collective wave function and is directly responsible for the computational tractability of Sp$(3,R)$.

It must of course be recognized that Sp$(3,R)$, like SU(3), is a spectrum-generating algebra rather than a symmetry algebra. Thus its usefulness lies in the fact that many Hamiltonians of interest, in particular collective Hamiltonians and the harmonic-oscillator shell-model Hamiltonian [cf. Eq. (7)], lie in its enveloping algebra.

Mathematically, Sp$(3,R)$ is the noncompact simple real Lie algebra of dimension 21 [sometimes denoted Sp$(6,R)$] whose complexification is C_3 in the Cartan classification. It is isomorphic to the algebra of *all* one-body bilinear products in the position and momentum observables. The representations of Sp$(3,R)$ have been determined.[11] But, what is more important, they are realizable in shell-model state space.

Recall that, in the shell-model realization of SU(3),[5] the generators can be expressed

$$C_{\alpha\beta}=\sum_{i=1}^{A} b_{\alpha i}{}^{\dagger} b_{\beta i} \quad (\alpha,\beta=x,y,z), \tag{5}$$

where $b_{\alpha i}{}^{\dagger}$ is the HO raising operator for nucleon i. The generators of Sp$(3,R)$ include, in addition to those of SU(3), the operators

$$A_{\alpha\beta}=\sum_{i=1}^{A} b_{\alpha i}{}^{\dagger} b_{\beta i}{}^{\dagger}, \quad B_{\alpha\beta}=\sum_{i=1}^{A} b_{\alpha i} b_{\beta i}. \tag{6}$$

Now it has been shown[11] that, if we start with a set of $0\hbar\omega$ SU(3) states $|(\lambda\mu)\times LM\rangle$ and generate the set of $2\hbar\omega$ states $A_{\alpha\beta}|(\lambda\mu)\times LM\rangle$, the $4\hbar\omega$ states $A_{\alpha\beta}A_{\gamma\delta}|(\lambda\mu)\times LM\rangle$, etc., the set of all states so generated carries an irreducible representation of Sp$(3,R)$. This building up process corresponds to starting from the basis states of a single $0\hbar\omega$ SU(3) representation and augmenting the space by successive application of $\hat{Q}$ and the monopole operator $\sum_i r_i{}^2$. Figures 1 and 2 illustrate the results obtained starting from the $(0,0)$ and $(8,0)$ SU(3) states, respectively.

It is interesting to note some similarities with the phenomenological model of Arima and Iachello,[12] based on the algebra SU(6), which has the same spectrum of states as the Sp$(3,R)$ model generated from the $(0,0)$ SU(3) representation. The identification results from the fact that the first excited level $(2,0)$ in the Sp$(3,R)$ representation is six dimensional and thus carries the fundamental representation of SU(6). However, the algebraic structures of the two models differ since Arima and Iachello take their excitation operators to be boson operators whereas the $A_{\alpha\beta}$ in the Sp$(3,R)$ model are bilinear products of boson operators and do not satisfy the oscillator commutation relations. Nevertheless, the remarkable successes of the SU(6) model[12,13] bode well for the microscopic Sp$(3,R)$ model.

The value of the Sp$(3,R)$ scheme is its simplicity. The states are very easy to calculate in

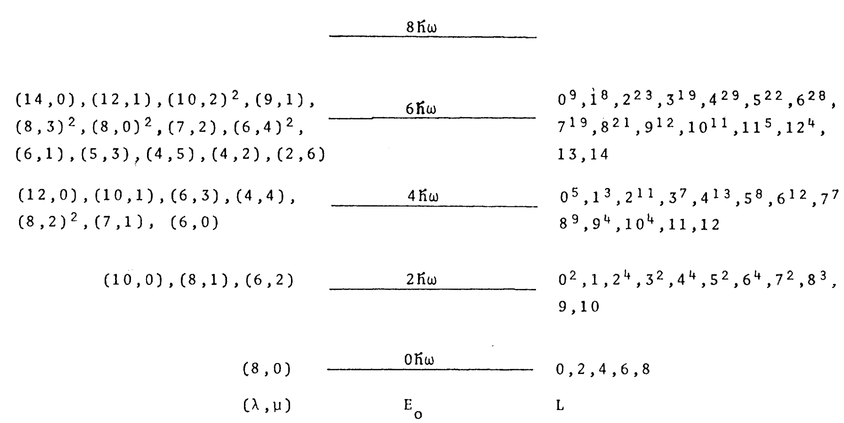

FIG. 2. SU(3) irreducible representations (λ, μ) spanning the Sp$(3,R)$ irreducible representation whose $0\hbar\omega$ subspace transforms as $(8,0)$.

terms of the shell model, since the $A_{\alpha\beta}$ are simple particle-hole operators. Moreover, since $A_{\alpha\beta}$ is an irreducible (2, 0) SU(3) tensor operator, its matrix elements are partially determined by the Wigner-Eckart theorem. Furthermore, one can truncate the basis at many levels since the number of states of a given angular momentum does not proliferate very rapidly with the number of shells, cf. Figs. 1 and 2. The total dimension of the $2\gamma\hbar\omega$ level is just $\binom{5+\gamma}{\gamma}$ times the dimension of the $0\hbar\omega$ level. The algebra also permits the straight-forward calculation of many observables of interest in the enveloping algebra of Sp(3,R), e.g., the shape operators $\hat{a}_2,\hat{a}_3$ and $E2$ transitions.

A rotational band spans an eigenspace of the invariants $\hat{a}_2$ and $\hat{a}_3$ of $[R^5]$SO(3). Thus, the calculation of rotational bands in the shell-model basis has the group theoretic meaning of determining the transformation from the SU(3) basis to the $[R^5]$SO(3) basis. In a truncated space, exact eigenstates of $\hat{a}_2$ and $\hat{a}_3$ do not exist. Nevertheless, $\hat{a}_2$ can be diagonalized in a truncated SU(3) basis and one can determine whether or not the resulting band approximately satisfies the rotational-model $[R^5]$SO(3) prediction for $E2$ transitions. Table I indicates the extent to which rotational bands can be constructed with only a few shells. (Note that, although the Sp(3,R) model space contains a certain small admixture of center-of-mass excited states, the operators $\hat{a}_2$ and $\hat{a}_3$ commute with the c.m. operators. The states represented in Table I are consequently entirely nonspurious.)

In reality we believe that physical states, especially in light nuclei, will not be pure rotational. In particular, we anticipate that shell effects will cause a more rapid fall off of the contributions from higher shells than pure rotational states would require. It is of interest therefore, to consider the spectrum that would emerge from diagonalization of the Hamiltonian $H = H_{\rm HO} + V(\beta, \gamma)$; e.g.,

$$H = H_{\rm HO} + \tfrac{1}{2}k(\beta^2 - {\beta_0}^2)^2 = H_{\rm HO} - x_1\hat{a}_2 + x_2{\hat{a}_2}^2, \qquad (7)$$

where $H_{\rm HO}$ is the independent-particle HO Hamiltonian. The spectrum for this Hamiltonian is easy to calculate since H is in enveloping algebra of Sp(3,R). The Sp(3,R) model can, of course, also be used simply as a means of generating basis states for the diagonalization of any Hamiltonian. The vital contribution of the Sp(3,R) model is then to provide a simple means of augmenting the $0\hbar\omega$ shell-model space to include the

TABLE I. The ratios $a_2(L)/a_2(0)$ for the eigenvalues of $\hat{a}_2$, $a_3(L)/a_3(0)$ for the expectation values of $\hat{a}_3$, and $B(E2; L\to L-2)/B(E2;2\to 0)$ for reduced $E2$ transition probabilities are given for the Sp(3,R) irreducible representation whose $0\hbar\omega$ subspace transforms as (8,0) under SU(3) for mass number $A = 20$. The results are given for the most deformed band in each of the subspaces, which include states up to $0\hbar\omega$, $2\hbar\omega$, and $4\hbar\omega$, respectively. The corresponding predictions of the rotational model (R.M.) are given for comparison.

	L	0	2	4	6	8
$0\hbar\omega$	$a_2(L)/a_2(0)$	1.0	0.97	0.93	0.85	0.75
	$a_3(L)/a_3(0)$	1.0	0.95	0.83	0.65	0.39
	$\frac{B(E2;L\to L-2)}{B(E2;2\to 0)}$			1.27	1.07	0.64
$2\hbar\omega$	$a_2(L)/a_2(0)$	1.0	0.98	0.94	0.87	0.78
	$a_3(L)/a_3(0)$	1.0	0.96	0.88	0.75	0.57
	$\frac{B(E2;L\to L-2)}{B(E2;2\to 0)}$			1.34	1.29	1.07
$4\hbar\omega$	$a_2(L)/a_2(0)$	1.0	0.99	0.95	0.90	0.82
	$\frac{B(E2;L\to L-2)}{B(E2;2\to 0)}$			1.37	1.39	1.28
R.M.	$a_2(L)/a_3(0)$	1.0	1.0	1.0	1.0	1.0
	$a_3(L)/a_3(0)$	1.0	1.0	1.0	1.0	1.0
	$\frac{B(E2;L\to L-2)}{B(E2;2\to 0)}$			1.43	1.57	1.65

states necessary for the development of collective vibrational and rotational motions. In this context it is important to stress that the algebra CM(3) < Sp(3,R) contains the generators of vibrational motions and of both irrotational and rigid-flow rotations and of all possible linear combinations.[10] Clearly many possibilities exist and it will be interesting to see what types of motion emerge from detailed microscopic calculations. Some of the possibilities will be investigated in a more complete presentation of the model to follow.

We are most grateful to Dr. J. P. Draayer for supplying us with a copy of his SU(3) Clebsch-Gordan and Racah routines,[14] and to Dr. S. S. M. Wong for valuable consultations.

[1]G. Rosensteel and D. J. Rowe, to be published.

[2]M. Baranger, J. Phys. (Paris), Colloq. 33, C5-61 (1972).

[3]H. Ui, Prog. Theor. Phys. 44, 153 (1970).

[4]O. L. Weaver, L. C. Biedenharn, and R. Y. Cusson, Ann. Phys. (N. Y.) 77, 250 (1973).
[5]J. P. Elliott, Proc. Roy. Soc. London, Ser. A 245, 128, 562 (1958).
[6]V. Bargmann and M. Moshinsky, Nucl. Phys. 23, 177 (1961).
[7]B. R. Judd, W. Miller, Jr., J. Patera, and P. Winternitz, J. Math. Phys. (N. Y.) 15, 1787 (1974).
[8]S. Goshen and H. J. Lipkin, Ann. Phys. (N. Y.) 6, 301 (1959); R. Y. Cusson, Nucl. Phys. A114, 289 (1968); S. Tomonaga, Prog. Theor. Phys. Jpn. 13, 467, 496 (1955).
[9]G. Rosensteel and D. J. Rowe, Ann. Phys. (N. Y.) 96, 1 (1976).
[10]P. Gulshani and D. J. Rowe, Can. J. Phys. 54, 970 (1976).
[11]G. Rosensteel and D. J. Rowe, to be published.
[12]A. Arima and F. Iachello, Phys. Rev. Lett. 35, 1069 (1975).
[13]H. Ogata, to be published.
[14]J. P. Draayer and Yoshimi Akiyama, Comput. Phys. Commun. 5, 405 (1973).

Scattering of Low-Energy Electrons by Excited Sodium Atoms Using a Photon and Electron Atomic Beam Recoil Technique*

N. D. Bhaskar,† B. Jaduszliwer, and B. Bederson
New York University, New York, New York 10003
(Received 15 September 1976)

A new method for measuring cross sections for the scattering of electrons by laser-excited atoms is described. It is a generalization of the atomic-beam recoil technique, taking advantage of the recoil of atoms during resonant photon interactions to spatially separate excited from nonexcited atoms. A preliminary value for the total cross section for the scattering of electrons by the $3^2P_{3/2}(m_F=3)$ state of sodium at 4.4 eV is presented.

We report here on preliminary measurements of the absolute total cross sections for the scattering of low-energy electrons by sodium in the $3^2P_{3/2}(m_F=3)$ state at 4.4 eV, using a novel laser excitation method. The method is a generalization of the atomic-beam recoil technique.[1-3] Advantage is taken of resonant photon recoil which spatially disperses the sodium atoms in proportion to the fractional time they spend in the excited state while undergoing collisions. In the experiment reported here, the atomic-beam recoil technique is used to determine absolute total cross sections. More generally, the double recoil technique described in this Letter appears to offer a new method for studying many types of excited-state scattering cross sections.

Detailed discussions of the recoil technique when dealing with ground-state atoms are presented in Refs. 1–3. Briefly, a narrow atomic beam is cross-fired by a beam of low-energy electrons. The atomic beam is velocity and spin-state selected before scattering, and can be spin analyzed after scattering. The spatial dispersion of the scattered atomic beam is measured by an analyzer-detector assembly which rotates about the scattering region. With use of suitable kinematic analysis, one can thereby obtain differential elastic and inelastic cross sections, including spin-exchange and spin-flip cross sections. In the scattering-out mode, that is, by measuring the ratio of atomic-beam intensities in the forward direction with and without the electron beam operating, one can obtain absolute, total cross sections.

Scattering from excited sodium atoms, prepared by cross-firing a sodium beam with a cw single-mode dye laser tuned to a nonoptically pumped hyperfine resonance line, has been the subject of a number of recent, innovative papers by Hertel and co-workers.[4-6] They have reported on differential superelastic and inelastic scattering and also have presented theoretical analyses of some of the physics of excited-state scattering. A discussion of the use of the recoil technique to obtain scattering amplitudes for the superelastic $3^2P_{3/2}\to 3^2S_{1/2}$ transition in sodium, including discussion of coherent effects in the scattering process, is presented by Bederson and Miller.[7]

A schematic diagram of the experimental setup is shown in Fig. 1. The atomic beam is cross-fired by mutually orthogonal electron and laser beams.[8] The atoms are optically excited in a region of uniform magnetic field (~ 700 G) oriented along the electron beam axis. This field serves to partially decouple the nuclear and atomic magnetic moments. A cylindrical lens is used to elongate the laser beam along the atomic beam axis.

The atomic beam is polarized and velocity selected by an offset Stern-Gerlach magnet, and can be spin-state analyzed by an E-H gradient