[7]J. B. Ozenne, thesis, Université de Paris-Sud, 1976 (unpublished).
[8]J. Schopman, P. G. Fournier, and J. Los, Physica (Utrecht) 63, 518 (1973), and references therein.
[9]P. H. Krupenie, J. Phys. Chem. Ref. Data 1, 423 (1972).
[10]N. H. F. Beebe, E. W. Thulstrup, and A. Andersen, J. Chem. Phys. 64, 2080 (1976).
[11]G. Herzberg, *Spectra of Diatomic Molecules* (Van Nostrand, New York, 1950).

Crossover Dimensions for Fully Developed Turbulence

U. Frisch, M. Lesieur, and P. L. Sulem
Centre National de la Recherche Scientifique, Observatoire de Nice, Nice, France
(Received 11 June 1976)

The analytic continuation of homogeneous, isotropic turbulence in nonintegral dimensions d is not realizable for $d<2$ because the energy spectrum generally becomes negative. Recent arguments, in favor of an $\frac{8}{3}$ crossover dimension below which the Kolmogorov 1941 theory is exact, are found questionable. The existence of a crossover dimension d_c (≈ 2.03) at which the direction of the energy cascade reverses is supported by a second-order closure calculation.

The existence for critical phenomena of crossover dimensions d_c above which the mean-field theory is exact, and all the recent work on $d_c - d$ expansions lead one naturally to ask the same questions for fully developed, incompressible turbulence, a problem generally believed to be at least as difficult as critical phenomena.[1-3] For homogeneous isotropic turbulence with Gaussian initial conditions and forcing, the energy spectrum $E(k)$ is, in principle, a well-defined functional of the forcing and initial spectra, which may be continued analytically in noninteger dimensions using, for example, the formal Reynolds number expansion.

(I) In a recent Letter, Forster, Nelson, and Stephen[4] analyze an infrared problem for $d\approx 2$ and find, for $d<2$, a nontrivial fixed point. We do believe that $d=2$ is a crossover dimension, but only in the sense that, for $d<2$, the homogeneous isotropic turbulence problem is no longer meaningful. Indeed, we have shown that for $d<2$ the energy spectrum, if initially positive, will usually become negative for arbitrarily small times. The essence of the proof is that (i) for small times the quasi-normal approximation is exact to $O(t^3)$ and (ii) the quasi-normal expression of the transfer has an emission coefficient, usually denoted[5-7] a_{kpq} which when $p=q$ becomes negative for $d<2$, whereas it is always positive for $d\geqslant 2$. The realizability [in the sense $E(k)\geqslant 0$] of the analytic continuation into nonintegral dimension $d>2$ remains open in general, but can be settled within the framework of second-order closures.

(II) An $\frac{8}{3}$ crossover dimension for intermittency corrections to the Kolmogorov 1941 theory (K41) has been proposed independently by Nelkin[2] and de Gennes.[8] The de Gennes derivation is parallel to the classical Ginsburg argument for critical phenomena: Denoting $\epsilon(x)$ the local dissipation $\nu(\nabla v)^2$ and assuming K41, we have for separations $\vec{s}$ lying in the inertial range, by dimensional analysis ($\bar{\epsilon}$ is mean energy dissipation),

$$\langle[\epsilon(\vec{s})-\bar{\epsilon}][\epsilon(\vec{0})-\bar{\epsilon}]\rangle \sim \nu^2\bar{\epsilon}^{4/3}s^{-8/3}.$$

Since the volume element goes like $s^{d-1}ds$, the mean square dissipation fluctuation has an infrared divergence if $d>\frac{8}{3}$.

Our point is that $\nu(\nabla v)^2$ is a dissipation-range quantity and that its inertial-range fluctuations are irrelevant. If, instead, we use the nonlinear expression of the rate of transfer of energy $\epsilon(x) \sim \vec{v}\cdot\nabla(v^2/2+p)$, we find

$$\langle[\epsilon(\vec{s})-\bar{\epsilon}][\epsilon(\vec{0})-\bar{\epsilon}]\rangle \sim \bar{\epsilon}^2 s^0,$$

and the crossover dimension becomes zero. Alternatively, we can argue that the inertial-range properties should not be changed if the dissipative term in the Navier-Stokes equation, namely $\nu\nabla^2$, is replaced by $-\nu(-\nabla^2)^\alpha$, where α is called dissipativity. Indeed it is shown[9] for a certain model equation that the inertial-range behavior is unaffected by the dissipativity in the limit of zero viscosity, as long as $\alpha>\alpha_{\rm cr}=(n-1)/2$, where n is the exponent of the inertial range. It is clear that in de Gennes's (and also Nelkin's) derivation, the crossover dimension is dissipativity-dependent (e.g., for $\alpha=2$ we find $d_c=\frac{20}{3}$ but for $\alpha=\alpha_{\rm cr}=\frac{1}{3}$, we recover $d_c=0$ again). All this suggests that K41 is invalid in any realizable ($d \geqslant 2$) dimensions.

(III) We turn to the question of the conservation laws for the inviscid unforced equations and the relationship with the direction of the energy cascase. For $d=3$, there is only one known positive-definite invariant, the energy $\int_0^\infty E(k)\,dk$; the energy cascade is to high wave numbers (direct) and presumably there is an ultraviolet catastrophe (a singularity appears at a finite time). For $d=2$, there is the additional conservation of the entropy $\int_0^\infty k^2E(k)\,dk$ leading to an inverse energy cascade in the infrared direction; furthermore, there is no singularity at a finite time.[10,11] What about $2<d<3$? We have checked that the entropy does not go over continuously into another conserved quantity. However, a continuity argument indicates that inverse transfer will still be favored for $d\approx 2$ since entropy is nearly conserved, at least for short times. But will this behavior persist, or will energy eventually leak through to high wave numbers?

In the present state of turbulence theory, it seems very difficult to decide this point directly for the analytic continuation of the primitive statistical Navier-Stokes equations. So, instead, we have used a realizable second-order closure introduced by Kraichnan, the test field model (TFM), which can easily be extended to noninteger $d>2$ [5,6,12]; the following results have been obtained both analytically and numerically (Reynolds number $R\approx 10^6$). For any $d\geq 2$, there is an inertial-range solution[13] with $E(k)\sim k^{-5/3}$, the energy cascade being in the infrared direction for $2\leq d<d_c$ ($d_c\approx 2.03$),[14] and in the ultraviolet direction for $d>d_c$. When energy is injected in a narrow wave number band near $k=1$, we obtain for $d>d_c$ (e.g., $d=2.05$), a stationary direct cascade and for $2<d<d_c$ (e.g., $d=2.02$) an inverse cascade such that the bottom of the $-\frac{5}{3}$ range moves to ever smaller wave numbers (Fig. 1). In the former case, total energy saturates and in the latter, it increases linearly at the injection rate (Fig. 2). Since in the numerical integrations, the dimension differs from 2 by only a few percent, a naive continuity argument suggests a quasi-two-dimensional behavior for about a hundred turnover times at forcing wave numbers (here of the order of one); this is why the TFM equations have been integrated up to $t=5000$. For $2\leq d<d_c$, in addition to the infrared $-\frac{5}{3}$ range, there appears an ultraviolet $-n(d)$ range with n varying from 3 to $\frac{5}{3}$ (Fig. 1). For $d=2$, this is the usual entropy inertial range[10,11] but for $2<d<d_c$, no conserved quantity cascades along this range. We have also considered the unforced equations with smooth initial data in the inviscid limit: For any $d>2$, the entropy becomes infinite at a finite time $t_*(d)$ proportional to $(d-2)^{-1}$ near $d=2$; for $2<d<d_c$, the energy is conserved indefinitely as it is for $d=2$, whereas for $d>d_c$, energy is dissipated at a finite rate after t_* (Fig. 3). This is consistent with the results on the direction of the energy cascade. The behavior of the TFM equations at d_c has not yet been investigated.

In conclusion, we stress that a second-order closure leads to an inverse cascade for $2<d<d_c$ in spite of the absence of an entropy-like con-

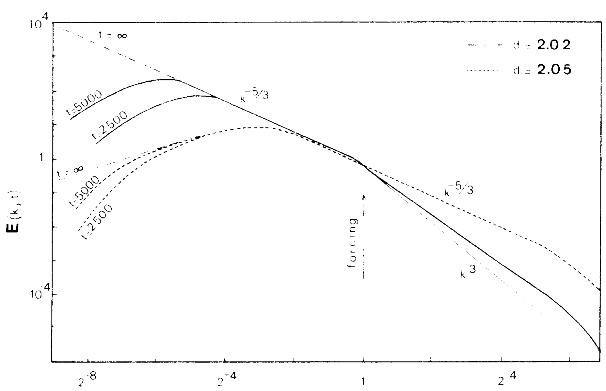

FIG. 1. Evolution of the energy spectrum below and above the crossover dimension $d_c\approx 2.03$. For $d=2.02$, there is an ultraviolet power-law range and a $-\frac{5}{3}$ infrared energy-inertial range proceeding to ever smaller wave numbers. For $d=2.05$, there is an ultraviolet energy inertial range which extends slightly into the infrared direction.

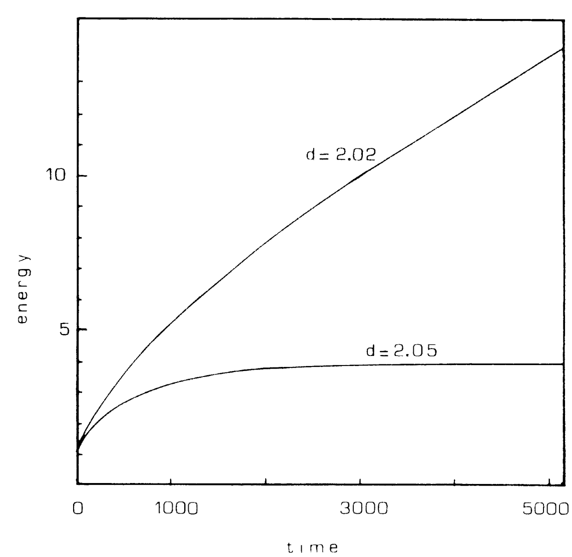

FIG. 2. Evolution of the energy with forcing below ($d=2.02$) and above ($d=2.05$) the crossover dimension.

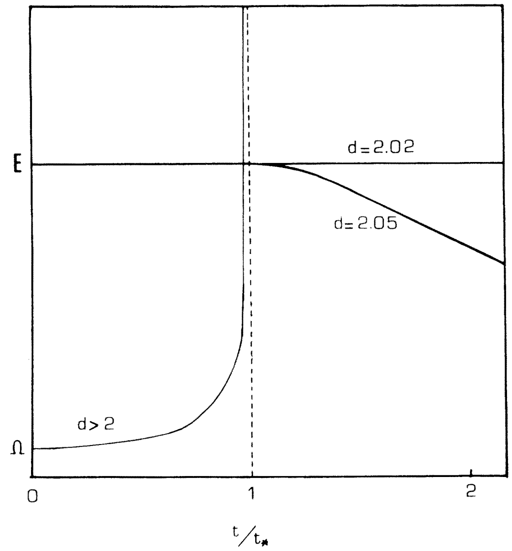

FIG. 3. Evolution of the energy E and entropy Ω without forcing in the limit of infinite Reynolds number. The catastrophe time $t_*(d)$ varies like $(d-2)^{-1}$ near $d=2$.

served quantity. This leads us to conjecture the existence of a similar crossover dimension for the primitive equations.[15]

We are indebted for discussion and suggestions to P. de Gennes, R. Kraichnan, P. Martin, M. Nelkin, M. Nelson, A. Pouquet, and H. Rose. We would also like to thank T. Bell and M. Nelkin for communication of a preprint that motivated part of this work.

[1]M. Nelkin, Phys. Rev. A 9, 388 (1974).

[2]M. Nelkin, Phys. Rev. A 11, 1737 (1975).

[3]P. C. Martin, J. Phys. (Paris), Colloq. 37, C1-57 (1976).

[4]D. Forster, D. Nelson, and M. Stephen, Phys. Rev. Lett. 36, 867 (1976).

[5]R. H. Kraichnan, J. Fluid Mech. 47, 513 (1971).

[6]D. C. Leslie, *Development in the Theory of Turbulence* (Clarendon, Oxford, England, 1973).

[7]S. A. Orszag, in *Statistical Theory of Turbulence*, edited by R. Balian (North-Holland, Amsterdam, 1976).

[8]P. de Gennes, in *Proceedings of the NATO Advanced Study Institute on Fluctuations, Instabilities and Phase Transition, Jeilo, Norway, 1974*, edited by T. Riste (Noordhoff, Leiden, 1975), Series B, p. 1.

[9]C. Bardos, P. Penel, U. Frisch, and P. L. Sulem, "Modified Dissipativity for a Nonlinear Equation Arising in Turbulence" (to be published).

[10]R. H. Kraichnan, Phys. Fluids 10, 1417 (1967).

[11]A. Pouquet, M. Lesieur, J. C. André, and C. Basdevant, J. Fluid Mech. 72, 305 (1975).

[12]P. L. Sulem, M. Lesieur, and U. Frisch, Ann. Geophys. 31, 487 (1975).

[13]Second-order closure does not yield any intermittency corrections.

[14]Another crossover dimension $d_c' \approx 2.208$ may be defined by the condition that for $d < d_c'$ high wave numbers in the $-\frac{5}{3}$ range act like a negative viscosity on small wave numbers as is done in a recent paper of R. H. Kraichnan, "Eddy-Viscosity in Two and Three Dimension" (to be published).

[15]Intermittency corrections to K41 should increase the $\frac{5}{3}$ exponent for $d > d_c$ and decrease it for $d < d_c$; see R. H. Kraichnan, J. Fluid Mech. 67, 155 (1975).

Auxiliary Heating of a θ-Pinch Plasma by Radial Magnetoacoustic Standing Waves

A. R. Jacobson, C. J. Buchenauer, J. N. Downing, and K. S. Thomas
University of California, Los Alamos Scientific Laboratory, Los Alamos, New Mexico 87545
(Received 7 July 1976)

Auxiliary heating of a linear θ-pinch plasma column by an externally driven radial magnetoacoustic oscillation has been experimentally investigated. The axial field of the θ pinch was modulated in time at the frequency of the plasma's fundamental radial magnetoacoustic oscillation. The dissipation in the plasma column was sufficient to transfer into the plasma at least 9% of the energy stored in the auxiliary capacitor bank used to drive the oscillation.

Driven radial magnetoacoustic waves are a potential form of auxiliary heating in θ-pinch devices following the initial implosion. We report here on preliminary experiments to investigate such heating. Small-amplitude $m=0$, $k=0$ natural radial oscillations were observed[1,2] and explained theoretically[3] on early θ pinches. In the results reported here, the main axial confining field was weakly modulated by an auxiliary bank at the resonance frequency of the fundamental radial oscillation. The amount of additional heating so obtained was inferred from the plasma temperature, at a later time after the column had become quiescent. This temperature was compared with that measured in the absence of resonant modulation of the confining field.