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Existence of a Phase Transition on a Classical Lattice Spin System with Three-Spin Interactions*

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It is shown that an appropriately defined magnetization has a zero thermal average above a critical temperature and a nonzero average below it for a specific lattice Hamiltonian with three-spin interactions in two or more dimensions. This work implies the existence of a phase transition in Reggeon field theory.

Lattice models for phase transitions have been very extensively studied in the last five decades. The Ising model provided an excellent laboratory for these studies; and it was first shown by Peierls that the Ising model in two or more dimensions has a phase transition at sufficiently low temperatures.[1] His argument was improved mathematically and somewhat generalized later, but no subsequent investigation went beyond pair interactions.[2,3]

I shall prove the existence of a phase transition on a classical square-lattice spin system defined by the Hamiltonian

$$H=-g\sum_{k,i}s_{k,i}s_{k,i+1}(s_{k+1,i}+s_{k+1,i+1}-s_{k-1,i}-s_{k-1,i+1})\equiv-g\sum_{k,i}[A_{ki}(s)+B_{ki}(s)-C_{ki}(s)-D_{ki}(s)], \tag{1}$$

where g is defined to be positive. The system is not symmetric in the first and second labels; their corresponding axes will be referred to as the time and space axes, respectively. The symmetries of the system are ST (combined spin and time reversal), P (space inversion), A_t and A_x (timelike and spacelike transitions, respectively, by one unit).

A lattice spin system similar to the one defined by (1) was shown to be equivalent to Reggeon field theory (RFT)[4] near the critical point.[5] However, that lattice model, as derived by Cardy and Sugar, had two spatial dimensions and an additional nearest-neighbor pair interaction.[5] The extension of the present results to two spatial dimensions is trivial; the present model is only discussed for reasons of simplicity. The pair interaction is probably irrelevant in the determination of the critical properties of the system; and the critical value of the corresponding coupling constant may well be zero.[5]

H has infinitely many ground states on an infinite lattice and a $(2^i\times 3)$-fold degenerate ground state on a torus with $3i$ rows and an even number of columns. This can be seen by considering the contribution of the internal interaction energy of a block of 2×2 spins. It is zero for the configurations $\begin{smallmatrix}+&+\\+&+\end{smallmatrix}$, $\begin{smallmatrix}+&-\\+&-\end{smallmatrix}$, and $\begin{smallmatrix}+&+\\-&+\end{smallmatrix}$; $-2g$, for $\begin{smallmatrix}-&+\\+&+\end{smallmatrix}$; and $-4g$, for $\begin{smallmatrix}+&+\\-&-\end{smallmatrix}$. All other configurations can be obtained by applying P (unchanged contribution), or S or T (contribution changes sign). A ground state should have a maximal number of blocks with contribution $-4g$; and the rest, with $-2g$. The only type of configuration satisfying this constraint is the one with alternating rows of up, down, and alternating spins. Spacelike shifts of individual rows of alternating spins and timelike shifts of the whole configuration generate a variety of ground states. The ground-state energy

per site in the thermodynamic limit (or on the finite torus) is $\epsilon = E_{g.r}/N = -8g/3$, or $\epsilon = -16g/3$ in two spatial dimensions.

I define a generalized magnetization, M as

$$M = \sum_{k,i}^{K,I} (\pm s_{k,i}), \quad KI = N, \tag{2}$$

where the plus sign is chosen if $k = 3n+1$ or if $k = 3n$ and $i = 2m+1$, where m and n are positive integers, and the negative sign, otherwise. The infinite lattice or toroidal thermal average of M is zero since $(1+A_t+A_t^2)(1+A_x)M = 0$. There is a ground state Γ_0^1, such that $M\Gamma_0^1 = N\Gamma_0^1$. All other ground states having rows of positive and negative spins identical with Γ_0^1 will be called type 1 (Γ^1). Ground states of type 2 (Γ^2) and type 3 (Γ^3) are obtained from those of type 1 by applying A_t and A_t^2, respectively.

I intend to show that there are numbers G_0 and G_1 such that the thermodynamic limit of $\langle M\rangle/N$ on a rectangular lattice is zero if $G = g/kT < G_0$ and different from zero if $G > G_1$. The boundary conditions on the $K \times I$ rectangular lattice are the following: All spins at the boundary give positive contributions to M. I shall then say that they belong to Γ_0^1.

First, I prove the absence of magnetization at high temperature. The thermal average of the magnetization can be written as[6]

$$\langle M\rangle = [\textstyle\sum_s MC(s)][\sum_s C(s)]^{-1}, \tag{3}$$

where the summation goes over all admissible configurations and

$$C(s) = \prod_{k,i}^{K,I} \{[1+XA_{ki}(s)][1+XB_{ki}(s)][1-XC_{ki}(s)][1-XD_{ki}(s)]\},$$

with $X = \tanh G$.[6]

Because of translation invariance the contribution of an individual spin in M to the coefficient of X^j in Eq. (3) disappears if the distance of that spin to the boundary of the lattice is larger than j. The contribution of spins near the boundary can be estimated to give

$$|\langle M\rangle| < 2(K+I)\textstyle\sum_j j(24X)^j, \tag{4}$$

where $2(K+I)j$ is the area of the boundary region, while 24^j is an upper bound on the number of jth order lattice diagrams[6] that can be drawn from a single site, since twelve different interaction terms can be attached to each site and each new interaction term creates two new spin sites. The counting includes overlapping disconnected diagrams as well.

The thermodynamic limit of the magnetization per site is zero if $G < \tanh^{-1}(\frac{1}{24}) = G_0$, since then the series in (4) converges and $(K+I)/KI$ disappears if $K, I \to \infty$.

I prove now the presence of magnetization at low temperature. The lattice can be divided into regions of three different types, R^k, $k = 1$, 2, and 3 (inside which the spins belong to ground states Γ^k, for every configuration). The boundaries of neighboring regions are not defined everywhere uniquely; some individual spins can be joined to either region at the boundaries. I shall make use of this arbitrariness in the course of the proof.

The boundary conditions imply that the ground-state region R_0^1 completely surrounds the bounded (maybe multiply connected) regions, R_k, that are themselves unions of one or more regions of different types. In what follows, I shall denote these regions with R. The first thing we need is to have a lower bound on the contribution of R to ΔE, the excitation energy of the given configuration. Such a lower bound is given by the following theorem: Let the total length of the vertical and horizontal boundaries of R be V and H, respectively. Then ΔE_R, the contribution of this region to the excitation energy, satisfies

$$\Delta E_R \geq 8gV/3, \quad \Delta E_R \geq 8gH/3. \tag{5}$$

The phrase length always refers to that of the shorter boundary if, because of the above mentioned arbitrariness, the boundaries can be changed and no intersection of boundaries is involved. The detailed proof of this theorem will be given elsewhere.[7] The proof is based on the observation that a block of four rows and two neighboring columns of spins has exactly a contribution of $-8g$ to the total energy in a ground-state configuration. The proof proceeds through the investigation of the contribution of pairs of columns and groups of four rows (intersecting R) to ΔE. The length of a very simple type of boundary (the only one, represented by an open line) that intersects an alternating row and merely changes its phase (two down or up spins side by side) can be included in V in Eq. (5).

A direct consequence of the theorem is the inequality

$$E_R \geq 4gB/3, \tag{6}$$

where $B = V + H$ is the total length of the boundary.

We can get a lower bound on m^α/N, the eigenvalue of M/N on a configuration α, as follows:

$$m^\alpha/N \geq 1 - \sum_R T(R) R X_R{}^\alpha/N, \tag{7}$$

where $T(R)$ is the area of a region R, while $X_R{}^\alpha$ is 1 if the region R appears in configuration α and 0 otherwise. The summation extends over all possible regions R. The thermal average of Eq. (7) contains $\langle X_R \rangle$, the probability that a random configuration contains R.

The boundary of R may be disconnected. Regions for which the connected part of the boundary, B, that contains the outer boundary is B_0 will be denoted by $R(B_0)$. We can give an upper bound on the expectation value of the sum in Eq. (7) as follows:

$$\begin{aligned} S &= \sum_R T(R) \langle X_R \rangle \\ &= \sum_{B_0} T(R(B_0))\, n/d \\ &\leq \sum_{B_0} T(R(B_0))\, \exp[-\beta E(B_0)/2], \end{aligned} \tag{8}$$

where the summation over B_0 means a summation over all possible connected boundaries B_0; $n = \sum' \exp(-\beta E)$, a sum over configurations containing one of the regions $R(B_0)$; and $d = \sum \exp(-\beta E)$, a sum over every configuration. $E(B_0)$ is the excitation energy, generated by B_0 only. Since the equality in Eq. (8) is trivial I shall only show the validity of the inequality below.

Every term in n can be factorized into $\exp[-\beta \times E(B_0)]$, a multiplier corresponding to the excitation energies due to the boundaries outside of B_0, and a multiplier corresponding to $B - B_0$. Since we wish to obtain an upper bound on S, we can neglect all contributions to d that correspond to configurations with boundaries intercepting B_0. As a result we can cancel the contribution of boundaries outside of B_0 in n/d. We can also cancel (up to some degree) the contributions coming from $B - B_0$ by finding appropriate contributions to d for every possible choice of $B - B_0$.

First of all we can write

$$B - B_0 = \cup_i B_i,$$

where B_i are the individual (maybe disconnected) components of $B - B_0$ that are surrounded by the same connected region. Each of the boundaries B_i can be labeled by an additional upper index indicating the type of the region that surrounds it. In each term of n the multiplier corresponding to $B - B_0$ can be factorized into multipliers corresponding to B_i, $i = 1, 2, \ldots$.

The contribution of $B_i{}^k$, $k = 1$, 2, or 3, to the excitation energy can be matched by the contribution of a boundary $\bar{B}_i{}^1$ that has exactly the same shape as $B_i{}^k$ but it is shifted up by one unit if $k = 2$, shifted down one unit if $k = 3$, and kept unchanged if $k = 1$. The orientation of spins inside the boundaries $B_i{}^k$ is kept unchanged during the translation (amounting to a cyclic permutation of the regions inside). The boundaries $\bar{B}_i{}^1$ would give the same contributions to ΔE if it were not for the possible difference of the phase of the alternating rows reaching $B_i{}^k$ and $\bar{B}_i{}^1$ from the outside. The phase of alternating rows that connects the two boundaries $\bar{B}_i{}^1$ and $\bar{B}_j{}^1$ is chosen in such a way that they reach the boundary on the right in the same phase as they reached the corresponding boundary $B_i{}^k$ or $B_j{}^m$.

If we change the phase of an alternating row reaching a boundary the change of the energy will be $8g$, 0, or $-8g$. This can be seen by considering the contributions of the two 2×2 blocks of spins that contain only the last spin from the alternating row in question to ΔE (other blocks near the alternating row will have unchanged contributions). Denoting the spin orientations in these blocks by ab, $c\delta$, and ef in the three rows (a, b, c, δ, e, $f = +1$ or -1) and subtracting the contributions to the energy for $\delta = +1$ and $\delta = -1$, we obtain $2[(c-e)(c-f) - (c-a)(c-b)]g$ which is always equal to $8g$, 0, or $-8g$.

The contributions that I keep in d will be that of a configuration built up from the boundaries $\bar{B}_i{}^1$ (no B_0) and those obtained from this configuration by horizontal independent shifting (by one unit to the right) of the individual boundaries $\bar{B}_i{}^1$ (the spins inside the shifted boundaries, $\bar{D}_i{}^1$, are kept unchanged). The phase of the alternating rows that connect two regions will be chosen to stay unchanged at the boundary on the right-hand side even after these shifts. Two boundaries can never collide after the horizontal shifts, since the distance of those boundaries has to be at least two units (neighboring boundaries $B_i{}^k$ and $B_j{}^m$ are separated by a portion of B_0 since $k \neq m$). The collision of two regions $\bar{B}_i$ and $\bar{B}_j$ in the vertical direction can also be avoided if we consider that this would only be possible if two boundaries $B_i{}^3$ and $B_j{}^2$ are separated by the minimal two units so that a stretch of B_0 lies in between them. However, by listing all the possibilities, one can see that the arbitrariness in the definition of these boundaries may be used to connect at least one of them to B_0. In this case only the contribution of one of the boundaries $\bar{B}_i$ appears in the denominator; and the collision is avoided.

We can write down the following inequalities at this point

$$\frac{n}{d} \leq \exp[-\beta E(B_0)] \prod_i \frac{\exp[-\beta E(B_i)]}{\exp[-\beta E(\bar{B}_i)] + \exp[-\beta E(\bar{D}_i)]} \leq \exp[-\beta E(B_0)] \prod_i \frac{1}{\exp(-8c_i g\beta) + \exp(-8d_i g\beta)}, \quad (9)$$

where $8c_i g$ and $8d_i g$ are the increases of the excitation energy due to the change of the phase of the alternating rows at the boundaries of $\bar{B}_i$ and $\bar{D}_i$, respectively. Obviously $c_i + d_i \leq n_i \leq V/3$, where n_i is the total number of alternating rows that connect the outer part of B_0 and B_i and the inner parts of B_0 and the left-hand side of B_i. Using Eq. (9) we can write

$$n/d \leq \exp[-\beta E(B_0) + 4\beta g V/3]. \quad (10)$$

Inequality (10) and the inequality $E(B_0) \geq 8gV/3$ (see theorem) imply the inequality (8).

Inequalities (7) and (8) combine immediately to give

$$\langle m \rangle / N \geq 1 - \tfrac{1}{2} \sum_{b=1}^{\infty} b^2 4^{3b} e^{-4Gb/3}, \quad (11)$$

where b is the length of B_0. The multiplier $b^2/4$ is an upper bound on $T(R(B_0))$. An upper bound on the number of boundaries with length b is $2N \times 4^{3b}$, where the multiplier N comes from the translation of the boundary over the lattice, the multiplier 2 takes care of the possible exchange of regions of type 2 and 3, and a multiplier 4^b gives an upper bound on the number of ways the phase of the alternating rows reaching B_0 from both sides can be chosen. Finally a multiplier 4^{2b} is included to account for the number of ways a boundary of length b can be drawn starting from a given point of the lattice. This number comes from the fact that at each step of the drawing of the boundary, we can continue in four different ways and a graph of third-order vertices can be drawn without lifting the pencil and going over each line at most twice.[7]

It is easy to see from Eq. (11) that if $G \geq G_1 = 3.87$ then the magnetization per site, $\langle M \rangle / N$, has a nonzero thermodynamic limit.

My final conclusion is that the phase transition exists in two or more dimensions. Since M is zero for a configuration in which all spins have the same expectation value, $\langle s_{k,i} \rangle$ has to vary as a function of the location below the critical temperature, implying the spontaneous breaking of translation invariance of the Hamiltonian. In field theoretic language, the vacuum expectation value of the field is zero for weak couplings, but above a critical coupling it becomes nonzero and coordinate dependent.

After the completion of the present work, I received a paper of Cardy,[8] who gave a field theoretic proof of the existence of a phase transition in RFT. However, the proof is based on perturbation theory. Since the perturbation expansion of RFT has been proved to be divergent,[9] this very original proof is seen to lose some of its power. The author is indebted to Dr. Cardy for the speedy communication of his results and to Dr. J. J. G. Scanio for discussions.

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