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Solitons in Nonuniform Media*

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Nonlinear wave propagation in inhomogeneous media is studied analytically in the model of the nonlinear Schrödinger equation. Exact solutions in the form of multisolitons, accelerated in the nonuniform medium, are obtained.

The propagation of waves in a dielectric medium is strongly affected by both the nonlinearity and the nonuniformity of the medium.[1] For example, the group velocity of a Langmuir wave packet changes with time because of the nonuniform plasma density $[n=n_0(1+x/L)]$,

$$\dot{v}_g=\frac{3\dot{k}_x v_e^{\,2}}{\omega_p}=\frac{3v_e^{\,2}}{\omega_p}\left.\frac{\partial\omega}{\partial x}\right|_k=\frac{3v_e^{\,2}}{L}.$$

In a uniform medium with a weak nonlinearity for which the dielectric function may be expandable in powers of $|E|^2$, where E is the amplitude, the cubic nonlinearity yields multisoliton solutions in the one-dimensional nonlinear Schrödinger equation,[2] each moving with constant velocity corresponding to the nonlinear state of modulational instability (or the oscillating two-stream instability).

In laser plasma irradiation experiments, the plasma is both inhomogeneous and nonlinear to the electromagnetic waves and the large-amplitude plasma waves. In the case of oblique incidence, density cavitation (solitons) near the resonance has been observed experimentally,[3] numerically,[4] and in computer simulations.[5] These solitons are observed to propagate slowly towards the underdense plasma. The large-amplitude wave packet modifies the plasma density profile by causing local density cavitation through the action of the ponderomotive force. It is of interest to ask whether the nonlinear density profile modification can significantly change the wave-packet propagation in an inhomogeneous plasma. More fundamental is the question whether the inhomogeneous plasma still supports solitons, particularly multisolitons which maintain their shapes and identities even after mutual collisions.[6] To address these questions, we present here exact, analytic solutions of the nonlinear Schrödinger equation in a linearly inhomogeneous plasma. Single solitons and multisolitons are found accelerated just as in the case of linear wave packets. Unlike the linear wave packets which spread in time, the solitons maintain their shapes and identities when moving around even upon emerging from collisions with other solitons.

The equation for the wave (Langmuir or electromagnetic) $\mathcal{E}(x,t)=E(x,t)\exp(-i\omega_0 t)$ with slowly varying $E(x,t)$ in a linearly inhomogeneous plasma with cubic nonlinearity is

$$i\frac{\partial E}{\partial t}+\frac{\partial^2 E}{\partial x^2}+(-2\alpha x+2|E|^2)E=0, \qquad (1)$$

where t is time expressed in units of ω_0^{-1}, and x is in units of $x_0=(3v_e^{\,2}/2\omega_p^{\,2})^{1/2}$ for the Langmuir wave and $x_0=c/\sqrt{2}\omega_p$ for the electromagnetic wave. The origin $x=0$ is the usual turning point where $\omega_p(x=0)=\omega_0$, $2\alpha=x_0/L$, and E is in units of $4(\pi n_0 T)^{1/2}$. The nonlinear term in this wave equation is expected to be important for (i) the linearly converted Langmuir wave at resonance for obliquely incident radiation,[7] (ii) Langmuir waves near the critical point where an electromagnetic wave parametrically decays into a plasma wave and an ion wave,[8] and (iii) Langmuir waves near the quarter-critical-density point where the incident electromagnetic wave decays into two plasma waves.[9]

In these cases, Eq. (1) describes the Langmuir wave after the pump (source) is switched off. For the electromagnetic wave, Eq. (1) holds near its turning point, such as the region near the critical density for normally incident radiation,[5] or the region near the quarter critical density for the Raman[9] scattered wave.

A one-soliton solution of Eq. (1) can easily be obtained by assuming $E=A(x,t)\exp[i\varphi(x,t)]$ with A and φ both real. The resulting equations for A and φ are

$$(A^2)_t+2(A^2\varphi_x)_x=0,$$
$$-[2\alpha x+(\varphi_t+\varphi_x{}^2)]A+A_{xx}+2A^3=0. \tag{2}$$

Anticipating solutions having a time-dependent velocity $v_g=4\xi-4\alpha t$ because of the inhomogeneity, with 4ξ the initial velocity, we assume that $A(x,t)$ is a function with argument $2\eta(x+2\alpha t^2-4\xi t-x_0)$, where x_0 is the initial position of the wave packet. Equation (2) then yields the solution for the phase φ and amplitude A,

$$\varphi=2(\xi-\alpha t)x-4[\tfrac{1}{3}\alpha^2t^3-\alpha\xi t^2+(\xi^2-\eta^2)t]+\varphi_0,$$
$$A=2\eta\,\mathrm{sech}\,2\eta(x+2\alpha t^2-4\xi t-x_0), \tag{3}$$

valid for $\partial\varphi/\partial t\ll 1$. For this solution, we may define a generalized, time-dependent wave number $k\equiv\partial\varphi/\partial x=2(\xi-\alpha t)$ and frequency shift $\Omega\equiv-\partial\varphi/\partial t=2\alpha x+4[(\xi-\alpha t)^2-\eta^2]=2\alpha x+k^2-4\eta^2$. Then, remarkably, the usual Hamiltonian equations for the wave-packet propagation in a nonuniform medium remain valid for the soliton,

$$\frac{dk}{dt}=-\frac{\partial\Omega}{\partial x}\bigg|_k,\quad v_g=\frac{dx}{dt}=\frac{\partial\Omega}{\partial k}\bigg|_x=4(\xi-\alpha t). \tag{4}$$

The frequency shift Ω remains to have the meaning of a Hamiltonian even for this highly complicated, nonlinear wave packet, and

$$\frac{d\Omega}{dt}=\frac{\partial\Omega}{\partial k}\bigg|_x\frac{dk}{dt}+\frac{\partial\Omega}{\partial x}\bigg|_k\frac{dx}{dt}=0$$

as in the usual stationary linear medium.

If $\xi>0$, the soliton travels toward the overdense region initially. It suffers a deceleration 4α, until $t=\xi/\alpha$, when it reaches its turning point $x_T=x_0+2\xi/\alpha$ and the velocity changes sign. The soliton is then reflected and accelerated toward the underdense region. The distance of possible penetration into the overdense region depends on the initial velocity 4ξ.

In order to solve the complete time evolution problem for Eq. (1), or in particular, to find the N-soliton solutions, we introduce the inverse-scattering transform[6] by transforming our unknown function $E(x,t)$ into a set of scattering data $[C_k,\ \zeta_k,\ \rho(\xi)$ defined below$]$ via the linear scattering problem[2,10] (eigenvalue problem):

$$V_{1x}+i\zeta V_1=E(x,t)V_2,$$
$$V_{2x}-i\zeta V_2=-E^*(x,t)V_1, \tag{5}$$

where V_1 and V_2 are components of a column vector wave function v, ζ is the eigenvalue, and $E(x,t)$ serves as the potential function of the scattering problem and is assumed to approach zero as $|x|\to\infty$. The time variable t plays only the role of a parameter in the direct scattering problem (5). To determine the time dependence of the wave function v, we assume

$$V_{1t}=AV_1+BV_2,\quad V_{2t}=CV_1-AV_2, \tag{6}$$

with $A=2i(i\zeta)^2+i|E|^2-i\alpha x$, $B=-2iE(i\zeta)+iE_x$, $C=2iE^*(i\zeta)+iE_x{}^*$, and $\zeta_t=\alpha$. Then the compatibility condition[10] for Eqs. (5) and (6), i.e., differentiating Eq. (5) with respect to x and Eq. (6) with respect to t and demanding $V_{xt}=V_{tx}$, yields precisely our basic Eq. (1). We note that the major difference introduced by the inhomogeneity is that the eigenvalues ζ are no longer constants but depend linearly on time. By allowing the time-varying eigenvalue, we have therefore greatly enlarged the set of exactly solvable nonlinear time-evolution equations.

The procedure for the inverse-scattering method is the following: (i) For a given initial profile $E(x,0)$, find the eigenvalues for both the continuum, ξ, and bound states (discrete spectrum), $\zeta_k=\xi_k+i\eta_k$. Then solve the scattering problem to find the reflection coefficient $\rho(\xi)$ and a set of constant coefficients C_k for the normalization of bound-state wave functions. These constitute the "initial scattering data." (ii) Solve Eq. (6) to find the time evolution of the eigenfunctions and eigenvalues and therefore the time dependence of the scattering data. Since these data do not depend on x and also $E(x,t)\to 0$ as $|x|\infty$, we can easily figure out the time dependence of the scattering data by going to the limit $|x|\to\infty$.[2] These are

$$\rho(\xi,t)=\rho(\xi,0)\exp\{(4i/3\alpha)[(\xi+\alpha t)^3-\xi^3]\}, \tag{7}$$

$$C_k(\zeta_k,t)=C_k(0)\exp\{(4i/3\alpha)[(\zeta_k+\alpha t)^3-\zeta_k{}^3]\}. \tag{8}$$

(iii) From these scattering data at time t, reconstruct the "potential" $E(x,t)$ by the inverse-scattering method. In this way, the nonlinear evolution equation Eq. (1) is solved for the time evolution of the initial profile $E(x,0)$. The solution is given by[2,10]

$$E(x,t)=-2K_1(x,x;t) \tag{9}$$

obtained by solving the Gelfand-Levitan-Marchenko (GLM) equations[2,10]:

$$K_1(x,y)=F^*(x+y)+\int_x^\infty K_2{}^*(x,z)F^*(z+y)\,dz,\quad K_2(x,y)=-\int_x^\infty K_1(x,z)F(z+y)\,dz, \tag{10}$$

where

$$F(x,t)=\sum_k^N C_k(t)\exp[i\zeta_k(t)x]+(2\pi)^{-1}\int_{-\infty}^{\infty}\rho(\xi,t)\exp[i\xi(t)x]d\xi \tag{11}$$

is constructed from the scattering data at time t, and $dK_2(x,x;t)/dx=\frac{1}{2}|E(x,t)|^2$. Because these are linear integral equations, solutions can be more easily obtained.

We will now discuss the solutions of Eq. (2):

(i) $\rho(\xi)=0$, N=*finite*.—In this case, we get N-soliton solutions. For clarity, we can write Eq. (11) in an abbreviated form,

$$F(x,t)\equiv\sum_k^N \overline{C}_k\exp(i\lambda_k x)$$

with $\overline{C}_k\equiv C_k\exp\{4i/3\alpha[(\zeta_k+\alpha t)^3-\zeta_k{}^3]\}$ and $\lambda_k\equiv\zeta_k+\alpha t$. The GLM equation can be solved by linearly expanding

$$K_j(x,y)=\sum_k^N K_{jk}(x)\exp(-i\lambda_k{}^*y),\quad j=1,2, \tag{12}$$

to obtain

$$K_{2k}{}^*(x)\exp(i\lambda_k x)=-i\sum_{m=1}^N\frac{\overline{C}_k\exp(2i\lambda_k x)}{\lambda_k-\lambda_m{}^*}K_{1m}(x)\exp(-i\lambda_m{}^*x),$$

$$K_{1k}(x)\exp(-i\lambda_k{}^*x)=\overline{C}_k{}^*\exp(-i\lambda_k{}^*x)+i\sum_{m=1}^N\frac{\overline{C}_k\exp(2i\lambda_k x)}{\lambda_m-\lambda_k{}^*}K_{2m}(x)\exp(i\lambda_m x). \tag{13}$$

These algebraic equations can be solved by the method of determinants to obtain the N-soliton solutions. The one-soliton solutions are especially simple. Setting $N=1$ in Eq. (12), we get

$$E(x,t)=-2K_1(x,x)=-\frac{2\overline{c}^*\exp(-2i\lambda^*x)}{1+(\overline{c}\overline{c}^*/4\eta^2)e^{-4\eta x}}=Ae^{i\varphi}, \tag{14}$$

yielding exactly Eq. (3) with $-\xi$. We notice that the real part of the eigenvalue ξ is responsible for the initial speed of the soliton, while the imaginary part of the eigenvalue η is responsible for the height and width of the soliton. In case of N solitons, the solutions are similar to those found in a homogeneous plasma,[2] differing only in that they are now accelerated. When far apart from each other, they are essentially a superposition of one-soliton solutions given by Eq. (3). Upon collision with each other, they emerge with their identities unaltered except for a minor shift in position. Sometimes, all eigenvalues or a subset of eigenvalues have identical real parts, corresponding to solitons moving together with the same initial velocity. Because they experience the same acceleration, they remain together forming a bound state called breathers. A breather containing many solitons will in general have a symmetrical irregular shape oscillating in time periodically. Under perturbation to the equation, the breather will break up and release all its solitons. An initial wave profile with a simple linear phase function like $E(x,0)=B(x)e^{ikx}$ with $B(x)$ real will develop into a breather. The direct scattering analysis shows[11] that solitons from this particular profile travel together with initial speed $2k$. Furthermore, if $B(x)=B\,\mathrm{sech}\,ax$, a typical pulse profile, then the direct scattering problem is exactly solvable. The condition for N solitons from this profile was shown by Satsuma and Yajima[11] to be $N< B/a< N+1$. They also showed that the fraction of initial pulse energy that goes into the nonsoliton part of the solution is $[(B-N)/B]^2$ and is negligibly small when $N\gg 1$.

(ii) *$N=0$, no bound state.*—For an initial profile with $\int|E|\,dx<=0.904$, Ablowitz *et al.*[10] showed that there is no bound state and therefore no soliton formation. This is also the threshold condition for nonlinear modulational stability in an inhomogeneous media. It is interesting to note that it remains the same in a linearly inhomogeneous media. The time evolution of solutions with general $\rho(\xi)$ is difficult to calculate exactly. We can nonetheless carry out an asymptotic analysis to find its long-time behavior. Using the saddle-point approximation,[10] we find that

$$F(x,t)=2\pi^{-1}\int_{-\infty}^{\infty}d\xi\,\rho(\xi)\exp\{-4i/3\alpha[(\xi-\alpha t)^3-\xi^3]+i(\xi-\alpha t)x\}$$

approaches

$$\frac{1}{4(\pi t)^{1/2}}\rho\left(\frac{\alpha t}{2}-\frac{x}{8t}\right)\exp\left\{\frac{4i}{3\alpha}\left[\left(\frac{\alpha t}{2}+\frac{x}{8t}\right)^3+\left(\frac{\alpha t}{2}-\frac{x}{8t}\right)^3\right]-i\left(\frac{\alpha t}{2}+\frac{x}{8t}\right)\right\}\sim O(t^{-1/2}), \tag{15}$$

as $t\to\infty$ with fixed $x/8t-\alpha t/2$. The GLM equation then tells us that $E(x,t)$ also goes like $O(t^{-1/2})$ asymptotically. The wave packet therefore spreads out and decays as $t^{-1/2}$ while traveling with group velocity $x/t-4\alpha t$.

(iii) The most general profile contains both the continuous spectrum and discrete bound states. Exact analysis is difficult. But one can still have the following picture of the time evolution of the profile. It breaks up into many solitons (or breathers) plus a spreading wave packet. The solitons propagate with constant speeds and undergo position shifts only occasionally as a result of collisions with one another. They maintain their identities after collisions. On the other hand, the wave packet spreads and decays in time like $t^{-1/2}$ while traveling with group velocity $x/t-4\alpha t$. Therefore, after a long while, only solitons remain. They are ordered in space according to their velocities, the fastest one lying at the farthest end and moving steadily ever farther.

Associated with Eq. (1) there are also an infinite number of conservation laws, $I_n=\int_{-\infty}^{\infty}f_n(xt)\,dx$, $n=1,2,\ldots$, and $dI_n/dt=0$. Following Zakharov and Shabat,[2] we can construct them from the recursion formula

$$f_1=|E(x,t)|^2,$$

$$f_{n+1}=E\frac{d(f_n/E)}{dx}+\sum_{j+k=n}f_jf_k+2i\alpha tf_n,\qquad n=1,2,\ldots. \tag{16}$$

They are closely related to the existence of multisoliton solutions.

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Note added.—After we submitted our paper, Dr. Fred Tappert kindly informed us that Eq. (1) can be transformed to the usual nonlinear Schrödinger equation directly by letting $x\to x-2\alpha t^2$ and $E\to E\exp(-2i\alpha xt-4i\alpha t^3/3)$. However, we believe that solving Eq. (1) directly through the inverse-scattering method as we did here is more convenient and pedagogically helpful. We do not have to juggle around both the dependent and the independent variables to follow its time evolution.

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Generation of an Intense Ion Beam by a Pinched Relativistic Electron Beam

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The tightly pinched electron beam of a pulsed electron accelerator is used to generate an intense beam of ions. A foil anode and vacuum drift tube are used. The space-charge field of the pinched beam in the tube accelerates ions from the foil anode. Ion currents of 10 kA at a density of 5 kA/cm^2 with pulse length of 50 nsec are obtained using a 5 kJ, 450 kV, 3-Ω diode.

For a possible application of ion beams to pellet fusion, high current densities of ions on the target are necessary.[1] It is therefore desirable to investigate high-current-density ion beams, both for direct application to fusion and for basic studies of the interaction of such beams with matter.

Since the discovery of ion acceleration by electron beams,[2] acceleration in vacuum drift tubes has been studied by several investigators.[3] The reflex triode technique has been suggested by Humphries, Lee, and Sudan[4] using either a real or virtual second cathode, and by Creedon and co-workers[5] using a thin anode foil. Other schemes of using anodes as ion sources have also been suggested.[6] Recently, high ion current densities were reported by Prono, Shearer, and Briggs[7] using the planar reflex diode technique.

The focal area of a tightly pinched electron beam seems to be a promising source of high intensity ion currents. With a foil anode, a high density of electrons is expected in this region on both the diode and the drift-tube sides of the anode. The space charge on the drift tube side is expected to accelerate ions generated in the anode plasma into the drift tube. In this work an intense electron beam, tightly pinched on a foil anode, was used to generate a high-current-density ion beam. Unlike the planar reflex diode technique, no external magnetic field was applied to prevent pinching. Rather, the diode was designed to produce a tight pinch. The measurements were performed directly on the ions accelerated down the drift tube.

The type of diode used is shown in Fig. 1. The brass cathode was 80 mm in diameter and was coated with Aquadag carbon. The anode was 40-μm-thick aluminum foil. The anode-cathode gap was 3.2 mm. The diode peak voltage was 450 kV and the peak current was 165 kA. The current rise time was 25 nsec, and the pulse width was 80 nsec. Under such conditions the beam tightly pinched, as was verified by x-ray pinhole photographs which were taken regularly. The drift tube was 15 cm in diameter, pumped down to 10^{-4} Torr. The drift-tube side of the anode was coated with 1 mg/cm^2 of CD_2. A target made of 0.3-mm-thick copper coated with 2 mg/cm^2 of CD_2 was centered on the drift tube axis at a distance of 4.5 cm from the anode. The radii of the coated portions on the anode and target were varied in the experiments. The properties of the ion beam were studied by analyzing the neutrons generated by reactions in the target and by studying

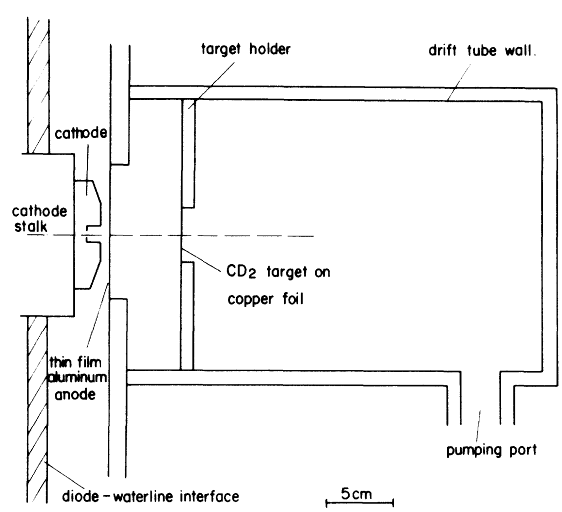

FIG. 1. Diode and target configurations for the ion acceleration experiments.