ployed here (see Fig. 3 caption), except that the 5.3-keV cobalt channel is 45% high for the streak camera. The relatively large error in this high-energy channel is likely related to the relatively weak recorded signal for cobalt, as observed in Fig. 2.

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Two-Dimensional Quasi-Linear Evolution of the Electron-Beam–Plasma Instability

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We report on the quasi-linear evolution of the electron-beam–plasma instability in a two-dimensional system. The numerical solutions of the basic equations show that a two-dimensional system evolves in a different way as compared with a one-dimensional system. After saturation the wave energy monotonically decreases with time while the width of its spectral distribution in k space narrows rapidly. In some cases, the system reaches a state which is modulationally unstable, and consequently cannot be described by the weak-turbulence theory.

In the problem of plasma heating, the turbulence excited by electron beams is of great importance. Various investigators have studied[1,2] the electron-beam–plasma interaction on the basis of the quasi-linear theory for the case when the turbulent oscillation spectrum is one-dimensional (wave vector $\vec{k}$ parallel to the motion of the beam). Such a model is appropriate when there is a magnetic field parallel to the beam in the plasma and when this field is strong enough to suppress oscillations that propagate at an angle to the beam axis. If there is a weak magnetic field in the plasma, so that $\omega_{pe} \gg \omega_{ce}$ (ω_{pe} and ω_{ce} are the plasma and the cyclotron frequencies of electrons, respectively), the turbulent spectrum becomes essentially three-dimensional. Such situations are met, e.g., in astrophysical problems or in the inertial confinement of plasmas. At present not much is yet known about the quasi-linear behavior of a three-dimensional turbulence excited by electron beams in a plasma. Attempts have been made to find at least certain general features of possible asymptotic states for the three-dimensional system.[3,4] However, these states seem to be too artificial and they have never been observed in experiments.[2] Thus the relaxation dynamics as well as the final state of the system remain unknown.

In this Letter we report on the two-dimensional quasi-linear evolution of the electron-beam–plasma instability. Since the problem exhibits axial symmetry with respect to the beam axis, the two-dimensional model is not restrictive[5] and was only chosen for convenience. The fundamental equations of the quasi-linear theory for the electron-beam–plasma interaction read[2]

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial \vec{v}} \cdot \pi \sum_{\vec{k}} \frac{\vec{k}\vec{k}}{k^2} I_{\vec{k}}\, \delta(\omega - \vec{k}\cdot\vec{v}) \cdot \frac{\partial f}{\partial \vec{v}}, \qquad (1)$$

$$\frac{\partial I_{\vec{k}}}{\partial t} = \frac{2\pi I_{\vec{k}}}{k^2 \partial\epsilon/\partial\omega} \int d^2 v\, \vec{k} \cdot \frac{\partial f}{\partial \vec{v}}\, \delta\,(\omega - \vec{k}\cdot\vec{v}), \qquad (2)$$

where $\omega = 1 + 3k^2/2$, $\partial\epsilon/\partial\omega = 2 + 3k^2$, f is the velocity distribution function for electrons, and $I_{\vec{k}}$ is the spectral distribution of the electrostatic field associated with the oscillations. Equations (1) and (2) are in dimensionless units; the units of time, space, velocity distribution function, and spectral distribution are, respectively, ω_{pe}^{-1}, λ_D, $m_e n/T_e$, and $4\pi n T_e$. Here m_e, T_e, and n are the electron mass, temperature, and density, respectively, and λ_D is the Debye length.

We have solved Eqs. (1) and (2) by the finite-element method.[6,7] The algorithm consists of the following stages. First, Eq. (1) is put into the Galerkin (weak) form, the natural (Neumann) boundary condition being imposed on the velocity distribution function. The approximate solution for f is then assumed to be a combination $\sum f_i e_i$ of the linear pyramid basis functions $e_i(v)$. In this manner, Eqs. (1) and (2) are transformed into a system of ordinary differential equations in time for the quantities f_i and $I_{\vec{k}}$. This system of equations is solved numerically by an implicit, four-level, time-centered, finite-difference scheme.

The accuracy of the solutions was verified by checking the variations of the integrals of motion of Eqs. (1) and (2) (particle, momentum, and energy densities) and by repeating the computations with different numbers of the basis functions and the wave modes, and with different time steps. For typical results presented in this Letter we used about 300 basis functions and about the same number of wave modes. The initial distribution function was chosen as

$$f(t=0)=[2\pi(1+\xi)]^{-1}\{\exp(-v^2/2)+\xi\exp(-\tfrac{1}{2}[(v_x-u)^2+v_y^2])\}.$$

The values of the parameters ξ and u were varied within certain ranges in such a manner that the condition for kinetic instability [the condition for Eqs. (1) and (2) to hold], viz. $\xi^{1/3}<1/u$, was satisfied. Typically, the values of ξ were between 10^{-5} and 5×10^{-2}, and those of u between 4 and 10. The initial spectral distribution $I_{\vec{k}}(t=0)$ was taken as a constant, the value of which was varied within the range 10^{-2}–10^{-6} times the maximal value of the spectral distribution reached at saturation.

In all sets of computations we have observed roughly two stages in the evolution of the system. They are determined by the time history of the total wave energy density defined as $W=\sum_{\vec{k}}W_{\vec{k}}=\sum_{\vec{k}}\frac{1}{2}I_{\vec{k}}\omega\partial\epsilon/\partial\omega$ (in units of nT_e). As seen from Fig. 1, the wave energy density first increases from an initial level to its maximum value W_{sat} within a time t_{sat}. After the saturation it monotonically decreases. The other quantities shown in Fig. 1 are the widths of the wave energy density in k space. They are defined as $\Delta k_\alpha=[\sum_{\vec{k}}(k_\alpha-\langle k_\alpha\rangle)^2W_{\vec{k}}/W]^{1/2}$, where $\langle k_\alpha\rangle=\sum_{\vec{k}}k_\alpha W_{\vec{k}}/W$. It can be seen that in the first stage they remain more or less constant, the value of Δk_y being about twice the value of Δk_x. During the second stage both widths narrow rapidly. Towards the end of the observation time the spectrum is so narrow that $(\Delta k)^2\lesssim W$ in some cases. This fact indicates that the weak-turbulence theory breaks down since the system becomes unstable with respect to the modulational instability of Vedenov and Rudakov.[8] Figure 2 displays the dependence of the quantity W_{sat} on the parameters ξ and u. Here the solid lines represent the saturation energy computed for the one-dimensional system, while the dots refer to the two-dimensional system. We notice that these values are very close to each other. Moreover, in all cases it turned out that the saturation time obeys the approximate formula $t_{sat}\approx4\ln(W_{sat}/W_0)/\gamma_{max}$, W_0 and γ_{max} being the initial wave energy density and the initial maximum growth rate, respectively. In Fig. 3 we present the spectral distribution of the field in the direction parallel to the beam axis ($k_y=0$) at a few instants in time. Here the plots (a) and (b) refer to the two-dimensional and one-dimensional systems, respectively. We see a distinct difference between these two cases. In the two-dimensional system the spectrum moves up several half-widths along the k_x axis whereas in the one-dimensional system its location remains the same. Furthermore, in the former case the spectrum is always narrower (at least by a factor of 2) than in the latter. Figure 4 shows two cross sections of the velocity distribution function at several times. It can be observed that in the first stage the distribution function flattens along the v_x axis in the resonant re-

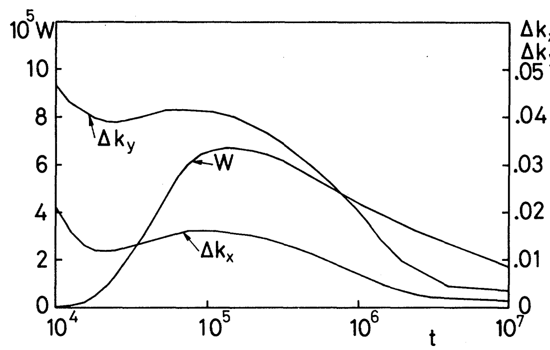

FIG. 1. Time history of W, Δk_x, and Δk_y. The initial conditions are $W_0=7\times10^{-8}$, $\xi=10^{-5}$, and $u=9$.

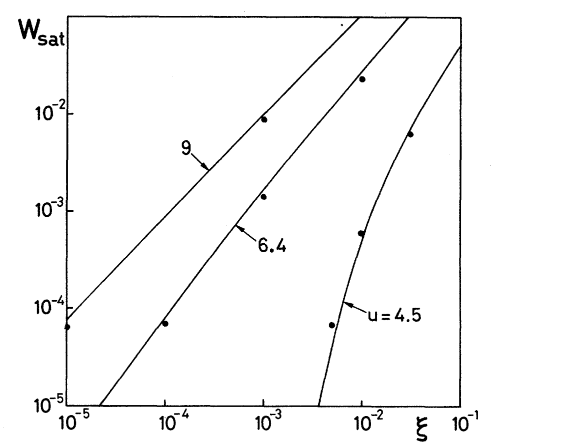

FIG. 2. Saturation wave energy W_{sat} versus parameter ξ for different values of u. The solid lines and the dots refer to the one-dimensional and two-dimensional systems, respectively.

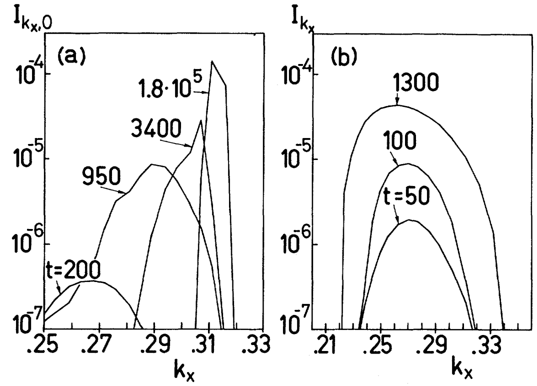

FIG. 3. Spectral distribution of the field at several instants in time (a) for the two-dimensional system at $k_y = 0$ and (b) for the one-dimensional system. The initial conditions are $\xi = 0.01$ and $u = 4.5$.

gion of the velocity space. In fact, such a flattening occurs at different heights for all cross sections along the lines $v_y = \text{const}$ in this region. At the same time the function broadens across these lines, the width being about twice the initial width of the beam. As the wave energy damps down during the second stage, a part of the bulk particles is heated in the direction parallel to the beam axis. Thus, a broad high-energy tail is formed on the distribution function.

During the course of our investigations[9] we were made aware[10] of the work of Ivanov, Sobolyeva, and Yushmanov.[11] With respect to the first stage of the quasi-linear evolution of the electron-beam–plasma instability in multidimensions their conclusions are similar to those presented in this Letter. However, the second stage, where, in our opinion, the effects of multidimensions play the most important role, was not treated in their paper. The possibility for the system under consideration to reach a state which cannot be described by the weak-turbulence theory, viz. a modulationally unstable state, may be regarded

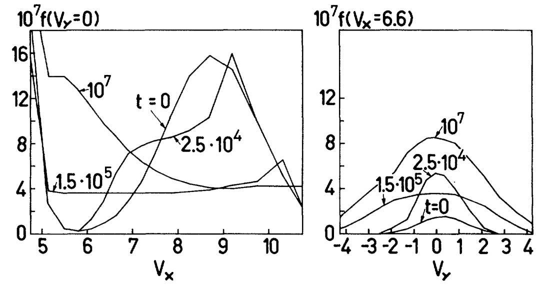

FIG. 4. Two cross sections of the velocity distribution function at several instants in time. The saturation time $t_{\text{sat}} = 1.5\times 10^5$, and $W_0 = 7\times 10^{-8}$.

as one of the main results of our treatment.

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