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Left-Degenerate Vacuum Metrics

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For all complex space-times in which the self-dual part of the Weyl tensor is algebraically degenerate, Einstein's vacuum equations are reduced to a single differential equation of the second order and second degree.

It is well known that Einstein's vacuum equations can be simplified considerably if the space-time admits a congruence of null shear-free geodesics,[1] or if the Weyl tensor is anti-self-dual.[2] Here we shall consider a broad class of complex metrics which includes both these as special cases.

We impose only one restriction on our space-time: that it admits a congruence of totally null surfaces. To describe them, we introduce the surface element

$$\Sigma_{ab} \equiv u_{,a} v_{,b} - v_{,a} u_{,b},$$

and the expansion form

$$\theta \equiv \theta_a dx^a \equiv \tfrac{1}{2}(u_{;a}{}^a\, dv - v_{;a}{}^a\, du),$$

where u and v are functionally independent scalars, constant on each surface. By a totally null surface we mean a differentiable two-space to which all tangent vectors are null. It follows that du and dv are null and mutually orthogonal. From this, one can easily prove that[3]

$$\Sigma_{ab;r}\Sigma^{rc} + \Sigma_{ab}\theta^c = 0.$$

In the special case $\theta = 0$, not merely is Σ covariantly constant on each surface, but the equations

$$x_{;r}{}^a \Sigma^{rb} = 0$$

have a tetrad of independent solutions. A totally null surface, therefore, is geodesic by definition, and plane if its expansion vector is zero.

The surface element is self-dual or anti-self-dual. We describe the congruence as left-handed in the first case, and right-handed in the second. A congruence of null shear-free geodesics is the intersection of a left-handed congruence of totally null surfaces with a right-handed one.[1] Here, of course, we are dealing with only one congruence. We take it to be left-handed.

For our purposes, the empty-space equations fall naturally into three classes: first, the three *surface equations*,

$$\Sigma_a{}^p R_{pq} \Sigma_b{}^q = 0;$$

second, the *central equations*, comprising $R = 0$ and the three remaining equations of

$$R_{aq}\Sigma_b{}^q = 0;$$

and third, the three *residual equations* of $R_{ab} = 0$. Since Σ is self-dual, null, and closed, the equation

$$\Sigma_{ap} C^{p}{}_{bqr} \Sigma^{rq} = 0$$

is an integrability condition for the surface equations.[4] It signifies that the self-dual part of the Weyl tensor is degenerate. Conversely, any left-degenerate vacuum space-time contains a congruence of left-handed null surfaces.[4] In the plane case, the surface equations are satisfied identically, and a stronger equation of left degeneracy,

$$C_{abqr} \Sigma^{rq} = 0,$$

is equivalent to $R = 0$.

Before setting to work on the field equations, we introduce complex coordinates specially adapted to the congruence. Two of them are u and v. The other two, x and y, are chosen so that

$$ds^2 = 2e^1e^2 + 2e^3e^4,$$

with

$$e^1 = \varphi^{-2}du, \quad e^2 = dx + \mathcal{P}\,du + \mathcal{R}\,dv,$$

$$e^3 = \psi^{-2}dv, \quad e^4 = dy + \mathcal{R}\,du + \mathcal{Q}\,dv.$$

On substituting into the integrability condition, we obtain

$$[\ln(\varphi/\psi)]_{xy} = 0,$$

which shows that a suitable transformation of the form $x \to X(u,v,x)$, $y \to Y(u,v,y)$, makes $\varphi = \psi$. (We remark, incidentally, that the two-form Σ/φ is now covariantly constant on each null surface.) The surface equations read

$$\varphi_{xx} = \varphi_{xy} = \varphi_{yy} = 0;$$

and we still have at our disposal coordinate transformations of the form

$$u \to U(u,v), \quad x \to [xV_v - yV_u + X(u,v)]W(u,v).$$

$$v \to V(u,v), \quad y \to [yU_u - xU_v + Y(u,v)]W(u,v).$$

We use them to put $\varphi = ay - bx + c$, with constant a, b, and c. The expansion form is now given by

$$\theta = \varphi(a\,du + b\,dv).$$

The metric belongs to the Plebanski-Schild class,[5]

$$ds^2 = ds_0{}^2 + 2\varphi^{-2}(\mathcal{P}\,du^2 + 2\mathcal{R}\,du\,dv + \mathcal{Q}\,dv^2),$$

where $ds_0{}^2$ is flat, while du and dv are null and orthogonal. In the special case

$$\Delta \equiv \varphi^{-2}(\mathcal{P}\mathcal{Q} - \mathcal{R}^2) = 0,$$

it reduces to the Kerr-Schild form.[6]

The central equations are more complicated. After some manipulation, one finds that the general solution contains three disposable functions: $\Pi(u,v,x,y)$, $f(u,v)$, and $g(u,v)$. It may be written as

$$\varphi^{-3}\mathcal{P} = \xi f - (\varphi^{-2}\Pi_y)_y,$$

$$\varphi^{-3}\mathcal{Q} = \eta g - (\varphi^{-2}\Pi_x)_x,$$

$$2\varphi^{-3}\mathcal{R} = \xi g + \eta f + (\varphi^{-2}\Pi_x)_y + (\varphi^{-2}\Pi_y)_x,$$

where $\xi = f$ and $\eta = g$ for $\theta \neq 0$, while $\xi = \frac{2}{3}x$ and $\eta = \frac{2}{3}y$ for $\theta = 0$. One can derive the second case as a limit of the first. The residual equations take the form

$$\Xi_{xx} = \Xi_{xy} = \Xi_{yy} = 0,$$

with the integral

$$2\Xi = y\alpha(u,v) - x\beta(u,v) + \gamma(u,v),$$

where Ξ is constructed from a, b, c, f, g, and Π. This is our one remaining field equation.

In the diverging case, we obtain $a = -b = 1$, $c = 0$, $f = g \equiv \frac{1}{2}\sqrt{\mu}$, by specializing the coordinates, and using the transformation

$$f \to f + 2ak, \quad g \to g + 2bk,$$

$$\Pi \to \Pi + k\varphi^3(fy - gx + k\varphi),$$

where k is a disposable function of u and v. We then find that

$$\Xi = \Delta + \varphi^{-2}(\Pi_x - \Pi_y)^2 + \tfrac{1}{2}\mu\varphi(\Pi_x + \Pi_y) - 3\mu\Pi + \varphi^{-1}(\Pi_{ux} + \Pi_{vy}) - \tfrac{1}{4}(x-y)(x\mu_u - y\mu_v).$$

We make μ constant and put $\alpha = \beta$ by specializing the coordinates further and using the transformation

$$\Pi \to \Pi + \tfrac{1}{6}\varphi^3 l(u,v), \quad \alpha \to \alpha + l_u + l_v, \quad \beta \to \beta - l_u - l_v.$$

The self-dual components of the Weyl tensor are given by[7]

$$C^{(5)} = C^{(4)} = 0, \quad C^{(3)} = -2\mu\varphi^3, \quad C^{(2)} = 2\beta\varphi^5,$$

$$C^{(1)} = 2\varphi^7[y\beta_u - x\beta_v - 2\beta(\Pi_x - \Pi_y) + (\partial_u + \partial_v)\{\tfrac{1}{2}\gamma - \mu\varphi^{-1/2}(\partial_x + \partial_y)\varphi^{-3/2}\Pi\}];$$

the anti-self-dual components, by

$$\tilde{C}^{(n)} = 2\varphi^3\partial_x{}^{5-n}\,\partial_y{}^{n-1}\Pi, \quad n = 1, \ldots, 5.$$

In the plane case, it is convenient to put $\varphi = 1$. We then have

$$\Xi = \Delta + \hat{\partial}_u \Pi_x + \hat{\partial}_v \Pi_y + \tfrac{1}{6}(y\hat{\partial}_u - x\hat{\partial}_v)(yf - xg),$$

where

$$\hat{\partial}_u \equiv \partial_u + f, \quad \hat{\partial}_v \equiv \partial_v + g.$$

We can make $\alpha = \beta = \gamma = 0$ by means of a transformation on Π; but this has the effect of introducing into the metric additional functions p, q, and r of u and v:

$$\mathcal{P} = -\Pi_{yy} + p + \tfrac{2}{3}xf, \quad \mathcal{Q} = -\Pi_{xx} + q + \tfrac{2}{3}yg, \quad \mathcal{R} = \Pi_{xy} + r + \tfrac{1}{3}(yf + xg).$$

When the field equation is satisfied, the Weyl tensor is given by

$$C^{(5)} = C^{(4)} = C^{(3)} = 0, \quad C^{(2)} = f_v - g_u, \quad C^{(1)} = (y\hat{\partial}_u - x\hat{\partial}_v)C^{(2)} - 2\hat{\partial}_v{}^2 p - 2\hat{\partial}_u{}^2 q + 4\hat{\partial}_{(u}\hat{\partial}_{v)} r,$$

and

$$\tilde{C}^{(n)} = 2\partial_x{}^{5-n}\partial_y{}^{n-1}\Pi, \quad n = 1, \ldots, 5.$$

Without loss of generality, we can make one function zero in each of the sets $\{f, g\}$ and $\{p, q, r\}$; if the Weyl tensor is left-null, we can make $f = g = 0$; if left-flat, $f = g = p = q = r = 0$.

One can deal with Einstein-Maxwell vacuum equations in much the same way, provided that one takes

$$\Sigma_{ab} F^{ab} = 0.$$

The surface equations are unaltered; Maxwell's equations give

$$F_{ab}\, dx^a \wedge dx^b = \epsilon(du\wedge dx + dv\wedge dy) + (\delta + x\epsilon_v - y\epsilon_u)du\wedge dv$$
$$+ \varphi^2[H_{xx} e^2\wedge e^3 + H_{yy} e^1\wedge e^4 + H_{xy}(e^1\wedge e^2 + e^4\wedge e^3)],$$

where the wedges indicate antisymmetrical tensor multiplication, ϵ and δ are disposable functions of u and v only, while H is subject to

$$H_{xu} + H_{yv} = \mathcal{P}H_{xx} + \mathcal{Q}H_{yy} + 2\mathcal{R}H_{xy};$$

and the remaining equations integrate in much the same way as in the purely gravitational system.[8] There is an interesting formal resemblance between the roles of Π in the last metric and H here.

The present work might well simplify the problem of finding real degenerate solutions in the case that has so far proved most refractory: that of twisting rays. Of greater interest, however, is the possibility of moving in the opposite direction: not specializing the anti-self-dual part of the Weyl tensor, but removing the present restriction on its self-dual part. This would presumably involve the introduction of a second Hertz function $\tilde{\Pi}$. Our conjecture is that Einstein's equations in the most general complex case could be reduced to a pair of differential equations of the second order and second degree.

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[1] I. Robinson and A. Trautman, Phys. Rev. Lett. 4, 431 (1960), and in *Conférence International sur les Théories Relativistes de la Gravitation, Jablonna, 1962,* edited by L. Infeld (Gautier-Villars, Paris, and PWN—Polish Scientific Publishers, Warsaw, 1964), pp. 107–114; R. P. Kerr, Phys. Rev. Lett. 11, 237 (1963).

[2] J. F. Plebanski, J. Math. Phys. (N. Y.) 16, 2395 (1975); J. D. Finley, III, and J. F. Plebanski, J. Math. Phys. (N. Y.) 17, 585 (1976).

[3] I. Robinson, J. Math. Phys. (N. Y.) 2, 290 (1961).

[4] J. F. Plebanski and S. Hacyan, J. Math. Phys. (N. Y.) 16, 2403 (1975); see also I. Robinson and A Schild, J. Math. Phys. (N. Y.) 4, 484 (1963).

[5] J. F. Plebanski and A. Schild, in Proceedings of the International Symposium on Mathematical Physics, Mexico City, Mexico, 5–8 January 1976 (unpublished), pp. 765–787, and to be published.

[6] G. C. Debney, R. P. Kerr, and A. Schild, J. Math. Phys. (N. Y.) 10, 1842 (1969).

[7] The expression for $C^{(1)}$ given here was derived by J. D. Finley, III, and A. Garcia.

[8] Details of this generalization will be given in a paper by A. Garcia and the present authors.