# Marginal-Stability Calculation of Electron Temperature Profiles in Tokamaks

Wallace M. Manheimer, K. R. Chu,* Edward Ott,† and Jay P. Boris
*Naval Research Laboratory, Washington, D. C. 20375*
(Received 8 March 1976)

Electron temperature profiles, turbulence levels, and anomalous transport are calculated for tokamak discharges using the hypothesis that the dissipative trapped-electron instability drives the plasma to marginal stability.

This Letter applies marginal-stability concepts to the calculation of tokamak profiles. The calculation is based on the idea that the tokamak temperature profile adjusts itself so that the dissipative trapped-electron mode (DTEM),[1-9] assumed to be the dominant transport mechanism, is at marginal stability. The energy from the external circuit goes into the plasma interior, and classical loss mechanisms near the outer edge (e.g., line radiation from impurities, deposition on the limiter, etc.) remove this energy. The dissipative trapped-electron instability is assumed to carry the energy from the interior to the edge. If the energy input at the center, energy outflow at the edge, and all other fluid properties of the plasma are known, the marginal-stability calculation gives the steady-state electron temperature and current profile. The turbulence level and the relation between anomalous transport coefficients can then be derived. While the energy input by the Ohmic heating current is well known, the details of the edge loss mechanisms are not well known. This introduces an uncertainty into this marginal-stability calculation which naturally appears as an uncertainty in the central temperature.

If some external mechanism drives the plasma toward an unstable state, an instability will be generated which then fights the external mechanism, driving the plasma back toward a stable state. If the instability is sufficiently strong, the plasma is maintained near marginal stability. If, on the other hand, some nonlinear mechanism[1,10-12] limits the fluctuations to a level smaller than that computed by our marginal-stability analysis, the resulting transport would not be sufficient to maintain the plasma at marginal stability. To evaluate this possibility requires nonlinear calculations of fluctuation levels with assumed unstable profiles. Here we adopt the marginal-stability hypothesis and explore its implications. Our calculations do in fact show that the mode amplitudes are small [Fig. 1(a)], thus lending credibility to the marginal-stability theory. Furthermore, the marginal-stability hypothesis also works well in other plasma situations such as predicting the behavior of resistive shocks[13] and mirror machines.[14] Recent particle simulations of both drift[15] waves and trapped-electron instabilities[16] also show that the initially unstable distribution function does relax to one which is marginally stable. During this relaxation process, the transport coefficient can be either greater[15] or less[16] than $\gamma/k^2$ depending on how easily marginal stability can be achieved.

In this calculation we use the simplest theory of the DTEM and so our model is corresponding-

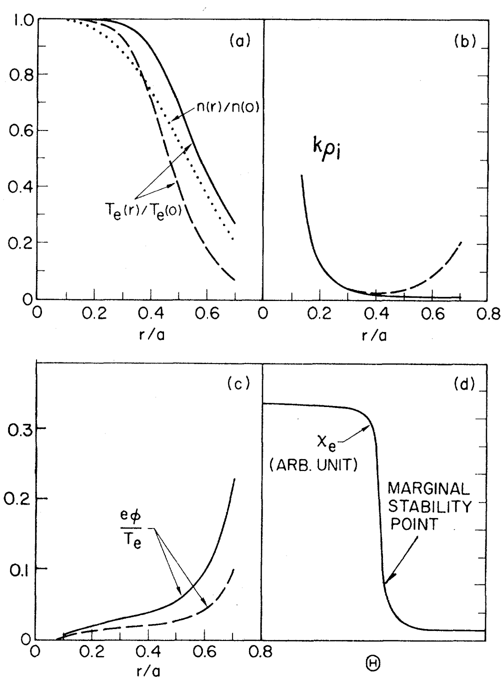


FIG. 1. Computed quantities for a Princeton large torus discharge assuming $n(r)/n(0) = \exp[-(r/0.6a)^3]$, with $T_e(0) = 3$ keV and $T_i(0) = 1.5$ keV and a toroidal field of 50 kG. (a) Electron temperature profiles for $q(a) = 3$ (solid line) and $q(a) = 5$ (dashed line); (b) values of $k\rho_i$, and (c) values of $e\varphi/T_e$ as a function of $r/a$; (d) an assumed form for the electron thermal conduction for use in a tokamak transport code.

ly limited, yet the results we obtain are encouraging. We intend to treat such factors as drift resonances[9] and other complicating effects on the mode, as well as edge energy-loss mechanisms, in a future paper. The growth rate of the DTEM is taken as

$$\frac{\gamma}{\omega_*}=\left(\frac{\epsilon}{2}\right)^{1/2}\eta_e\ \mathrm{Im}\left\langle\frac{3-(v/v_e)^2}{1+i(\nu_{ef}/\omega_*)}\right\rangle-\frac{\gamma_s}{\omega_*}, \tag{1}$$

where $\omega_*$ is the drift frequency taken equal to the wave frequency (valid for small $\epsilon=r/R$, and $k^2\rho_i^2 \ll 1$), $\eta_{e,i}=d\ln T_{e,i}(r)/d\ln n(r)$, angular brackets denote an energy average, and $\nu_{ef}=\nu(v^2)/\epsilon$, where $\nu(v^2)$ is the 90°-scattering collision frequency for an electron with speed $v$. In Eq. (1) $\gamma_s$ is the shear-induced damping rate[2-4];

$$\gamma_s/\omega_*=[1+2(T_i/T_e)]\Theta. \tag{2}$$

Here $\Theta$, the shear parameter, is the density-gradient scale length $L_n$ divided by the shear length $L_s=(r/Rq^2)^{-1}(\partial q/\partial r)^{-1}$. The tokamak safety factor is $q=rB_\zeta/RB_\theta$. Also we assume that $\eta_i=1$. We note that $\gamma_s$ is not precisely known for a tokamak since the mode structure is two-dimensional. We use here the standard expression for one-dimensional mode structure.

For the plasma to be at marginal stability, the shear must just stabilize the most unstable mode. It can be shown that the maximum value of $\mathrm{Im}\langle[3-(v^2/v_e^2)][1+i(v_{ef}/\omega_*)]^{-1}\rangle$ is 0.348 at $\omega/\nu_{ef}(v_e^2)=0.125$. Thus the marginal-stability condition relates $T_e$ to $q$:

$$\left(1+\frac{2T_i}{T_e}\right)\left(\frac{r}{R}\right)^{1/2}\frac{dq}{dr}=-0.25q^2T_e^{-1}\frac{dT_e}{dr}. \tag{3}$$

If the resistance is classical, $J_z(r)=J_z(0)[T_e(r)/T_e(0)]^{3/2}$. Then using Faraday's law and $q(0)=cB[2\pi RJ_z(0)]^{-1}$, we obtain another equation relating $q(r)$ and $T_e(r)$,

$$\frac{dq}{dr}=2q(r)r^{-1}\left[1-\left(\frac{T_e(r)}{T_e(0)}\right)^{3/2}\frac{q(r)}{q(0)}\right]. \tag{4}$$

We integrate Eqs. (3) and (4) for $q$ and $T_e$ outward to the limiter ($r=a$) starting with $r=\delta r$ and $q=1+\delta q$; $\delta r$ is chosen without loss of generality to be $0.1a$. Inside $r<0.1a$, with $q$ near 1, the energy transport is provided by internal-kink modes, not included in this treatment. [The need to invoke internal kinks near $r=0$ is reflected in our equations by the presence of a singularity of the coupled equations at $r=0$. One can prove from Eqs. (3) and (4) that the only solutions well behaved at $r=0$ are $q=\text{const}$, $T_e=\text{const}$.] The value of $\delta q$ then serves as an eigenvalue and is selected to give $q$ a particular value at the limiter. Notice that $q(a)$ is determined by the total plasma current and hence by the external circuit. Solutions for $T_e(r)/T_e(0)$ using typical Princeton large torus parameters are shown in Fig. 1(a). The eigenvalues $\delta q$ are chosen so that $q(a)=3$ and 5.

Notice that the central temperature $T_e(0)$ scales out of both Eqs. (3) and (4) so that Eqs. (3) and (4) are not sufficient to determine $T_e(0)$. This is to be expected, since $q$ and the growth rate in this model depend only on the shape of the temperature profile and not on the temperature itself. To determine the central temperature, one could exploit the fact that the dissipative trapped-electron instability transports energy from the center to an edge region of large classical losses. The condition which determines the central temperature is that the power loss from the edge equals the power input at the center. The power input due to the Ohmic heating decreases with $T_e$ as $T_e^{-3/2}$. The power loss from the edge region, while not easy to specify exactly, almost certainly increases with edge temperature [e.g., radiative loss, or electron energy loss to the limiter which scales as $n(a)T_e^{3/2}(a)$]. Therefore, the central temperature can be found by balancing input and output power. We will not pursue edge-loss mechanisms further here, but will simply assume a central temperature.

From quasilinear theory, the energy flux $Q_e$ and total particle flux $F_n$ are[11]

$$\begin{Bmatrix}Q_e\\F_n\end{Bmatrix}=\left(\frac{\epsilon}{8}\right)^{1/2}\sum_k\left(\frac{e\varphi(k)}{T_e}\right)^2\frac{cT_e}{eB}k\int_0^\infty 4\pi v^2\begin{Bmatrix}\frac{5}{6}mv^2\\1\end{Bmatrix}\frac{\omega_*\eta_e(\frac{1}{2}mv^2/T_e-\frac{3}{2})}{\nu_{ef}^2+\omega_*^2}\nu_{ef}f_0(v)\,dv. \tag{5}$$

At steady state, the equation for the energy flux is

$$\frac{d}{dr}Q_e+\frac{Q_e}{r}=\sigma^{-1}J^2(r)-\frac{n(T_e-T_i)}{\tau_{\mathrm{eq}}}, \tag{6}$$

where $\sigma$ is the conductivity of the plasma and $\tau_{\mathrm{eq}}$ is the electron-ion temperature equilibration time. Other bulk sources and sinks of energy (i.e., charge exchange, neutral beams, etc.) could easily be included in Eq. (6). The result would be a different value of $Q_e$, and therefore of $e\varphi/T_e$, and a different central electron temperature. However as long as $Q_e$, as solved for by Eq. (6), is everywhere positive (clearly there can be no steady state if $Q_e < 0$), the temperature profile $T_e(r)/T_e(0)$ would be unchanged.

To proceed, we specify all fluid parameters except the electron temperature profile and current profile. The results are shown in Fig. 1(a). Since the spectral energy is concentrated in the marginally unstable mode (whose $k$ value is given by $\omega_*/\nu_{ef} = 0.125$), one can solve for $e\varphi/T_e$ from (5). The results are shown in Figs. 1(b) and 1(c). With $e\varphi/T_e$ known, $F_n$ can be found from Eqs. (5). The energy confinement time [which scales as $T_e^{5/2}(0)$] $\tau_E = (IV)^{-1}\int \frac{3}{2} n(T_e + T_i)\, d^3r$ is 0.27 sec for the calculated temperature profile for $q(a) = 5$. The particle confinement time at radius $r$ is defined as $\tau_p = \int_0^r n(r')2\pi r'\, dr'/2\pi r F_n(r)$. $\tau_p(r)$ turns out to be a decreasing function of $r$. For instance $\tau_p(0.3a) = 1.2$ sec while $\tau_p(a) = 250$ msec.

Let us now examine the question of whether the temperature profiles we have calculated are in fact dynamically stable. Imagine that we perturb the temperature profile toward a stable state so that the instability is suddenly turned off. Then the plasma will heat for a time $dt$ and the amount of additional energy going into the electrons is proportional to $T_e^{3/2}(r)$. If the current drift velocity is larger than the sound speed,[17] so that only a small fraction of the Ohmic energy is lost to the ions by electron-ion temperature equilibration, it is a simple matter to show that the change in $T_e^{-1}\, dT_e/dr$ is proportional to $T_e^{1/2} n^{-2}(\eta_e - 2)\, dn/dr$. Thus if $\eta_e > 2$ the temperature profile steepens, while if $\eta_e < 2$, it broadens. The time for growth of the unstable waves is short compared to the current diffusion time. Thus the instability occurs while the current (i.e., shear) is frozen. Hence steepening the temperature gradient brings the instability back, while broadening it turns it off. Therefore, if $\eta_e > 2$, the temperature profiles we calculate are dynamically stable states, while if $\eta_e < 2$, the profiles are unstable, and never form. Notice from Fig. 1(a) that $\eta_e > 2$ for the case where $q(a) = 5$, but not for $q(a) = 3$. In the latter case the dissipative trapped-electron mode would not saturate by evolving toward marginal stability. It would either saturate at a higher level through other nonlinear mechanisms (and hence anomalous transport would be enhanced), and/or a new profile would try to form with a larger $q(a)$ (i.e., lower total current).

Let us close with a brief discussion of how both to find and to test the marginal-stability hypothesis with the use of tokamak transport codes.[18] To do so let the appropriate transport coefficient, for instance the electron thermal conductivity, depend on some parameter (for instance shear) in the way shown in Fig. 1(d). The lower part of the curve is the classical value, while the upper part to the left is some large value which comes from a nonlinear or turbulence theory. The sharp transition occurs at marginal stability.[19] The transport coefficient will not necessarily be given by the upper, turbulent value. Rather the profile can adjust itself to be near marginal stability. Then the transport coefficient will adjust itself to whatever point on the nearly vertical part of the curve it needs to maintain marginal stability.

We would like to acknowledge very useful conversations with Dr. M. N. Rosenbluth and Dr. F. W. Perkins. This work was supported by the U.S. Energy Research and Development Administration.

---

*Permanent address: Science Application, Inc., McLean, Va. 22101.

†Permanent address: Cornell University, Ithaca, N. Y. 14853.

[1]B. B. Kadomtsev and O. P. Pogutse, Dokl. Akad. Nauk SSSR 186, 553 (1969) [Sov. Phys. Dokl. 14, 470 (1969)].

[2]R. Z. Sagdeev and A. A. Galeev, Comments Plasma Phys. Controlled Fusion 1, 23 (1970).

[3]C. S. Liu, M. N. Rosenbluth, and W. M. Tang, Princeton Plasma Physics Laboratory Report No. MATT-1125 (to be published).

[4]L. D. Pearlstein and H. L. Beck, Phys. Rev. Lett. 23, 220 (1969).

[5]W. Horton, Jr., *et al.*, in *Proceedings of the Fifth International Conference on Plasma Physics and Controlled Nuclear Fusion Research, Tokyo, Japan, 1974* (International Atomic Energy Agency, Vienna, 1975), Paper No. CN-33/A14-3.

[6]P. C. Liewer, W. M. Manheimer, and W. M. Tang, Phys. Fluids 19, 276 (1976).

[7]K. B. Chu and W. H. Manheimer, to be published.

[8]K. T. Tsang and J. D. Callen, Oak Ridge National Laboratory Report No. ORNL-TM-4848 (to be published).

[9]W. M. Tang, P. H. Rutherford, H. P. Furth, and J. C. Adam, Phys. Rev. Lett. 35, 660 (1975).

[10]A. A. Galeev, in *Proceedings of the Third International Symposium on Toroidal Plasma Confinement, Garching, Germany, 1973* (Max-Planck-Institut für Plasmaphysik, Garching, Germany, 1973).

[11]W. M. Manheimer, K. R. Chu, E. Ott, J. P. Boris, and J. D. Callen, Nucl. Fusion 16, 203 (1976).

[12]E. Ott and W. M. Manheimer, Phys. Fluids 19, 1035 (1976).

[13]W. M. Manheimer and J. P. Boris, Phys. Rev. Lett. 28, 659 (1972).

[14]L. D. Pearlstein, D. E. Baldwin, and H. L. Berk, Bull. Am. Phys. Soc. 20, 1249 (1975).

[15]W. W. Lee and H. Okuda, Phys. Rev. Lett. 36, 870

(1976).
[16]Y. Matsuda and H. Okuda, Phys. Rev. Lett. 36, 474 (1976).

[17]Even though current velocity is greater than $c_s$, ion acoustic waves are stable since $T_i/T_e \sim 1$. Furthermore there is evidence that trapped particles can suppress current-driven instabilities. See V. Arunasalam, M. Okabayashi, R. J. Hawryluk, and S. Suckewer, Phys. Rev. Lett. 36, 726 (1976).

[18]D. F. Düchs, H. P. Furth, and P. H. Rutherford, in *Proceedings of the Fifth International Conference on Plasma Physics and Controlled Nuclear Fusion Research, Tokyo, Japan, 1975* (International Atomic Energy Agency, Vienna, 1975), Paper No. CN-28/C4.

[19]This is the basic approach being used in Naval Research Laboratory early-time debris air coupling codes. See, for instance, M. Lampe, W. M. Manheimer, and K. Papadopoulos, U. S. Naval Research Laboratory Memorandum Report No. 3076, 1975 (unpublished).

# Viscosity and Viscoelasticity of Two-Phase Systems Having Diffuse Interfaces*

Robert W. Hopper
*Department of Materials Science and Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139*

(Received 14 April 1976)

The equilibrium stability criterion for diffuse interfaces in a two-component solution with a miscibility gap requires that the interdiffusion flux vanish. If the system is continuously deformed, convective fluxes disrupt the equilibrium in the interface regions and induce a counter diffusive flux, which is dissipative and contributes to the apparent viscosity of the mixture. Chemical free energy is recoverably stored, causing viscoelastic phenomena. Both effects are significant.

The interfaces separating two phases in a two-component solution having a miscibility gap are often diffuse, the composition varying smoothly from phase to phase. More generally, diffuse interfaces are found (or are believed to exist) in a number of important types of multiphase systems, including liquid-gas mixtures near the critical point, phase-separated glasses, block copolymers, and various ordering systems. Such interfaces are described approximately by the theory of Cahn and Hilliard[1] (or some modification of it), in which the interface profile is determined by a variational principle minimizing the free energy, including a contribution from "gradient" energies. The basic concepts of the theory have been extended and appear to be confirmed experimentally.[2] If a system containing diffuse interfaces is continuously deformed, convective fluxes disrupt the thermodynamic equilibrium in the interface regions and induce a counter diffusive flux, which is dissipative and contributes to the apparent viscosity of the mixture. In addition, chemical free energy is recoverably stored, giving rise to a viscoelastic behavior. In the present paper, a theory of this effect is developed in outline and applied to a simple case.[2]

The theory is conceptually closely related to mode-mode coupling theories of the excess viscosity of solutions just above the consolute temperature.[2-6] The present theory provides an approach to the more complicated two-phase subcritical region. The theory is classical in the sense that the critical point singularities of the free energy are ignored; however, it is known[7,8] that these can be incorporated into the diffuse-interface theory upon which the present analysis is based. It seems, therefore, that the present treatment can be extended, without qualitative change, to include the singularities.

The following additional simplifying assumptions are made: The molar volume is independent of composition; the solution is isotropic when uniform; the composition dependence of the viscosity is ignored; and the temperature changes due to dissipation are negligible. The dimensions of the various quantities are defined in a footnote.[9] The theory is developed in detail only for a single flat interface and then applied naively to more complex situations.[2]

In gradient-energy theories,[1] the Helmholtz free energy is written as an integral over the volume of a free energy per unit volume of the nonuniform solution, $f^*(\vec{r})$, and that $f^*(\vec{r})$ depends only upon the local composition and all of its spatial derivaties. Expanding $f^*$ in the composition gradients, one eventually obtains[1] $f^*(\vec{r}) = [f(C) + \kappa \nabla C \cdot \nabla C]|_{\vec{r}}$, where $\kappa$ is a positive constant, and $f(C)$ is the free energy per unit volume of a uni-