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Solitons and the Motion of Helical Curves*

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Various nonlinear evolution equations that have been solved by the two-component inverse-scattering method are shown to be associated with the motion of certain simple types of helical curves. The linear eigenvalue problem that is usually postulated for solving these equations is found to follow *quite incidentally* from the standard intrinsic equations that govern the motion of space curves.

It has been shown by Hasimoto[1] that the intrinsic equations governing the curvature and torsion of an isolated thin vortex filament moving without stretching in an incompressible inviscid fluid may be reduced to a nonlinear Schrödinger equation. The single-soliton solution of this equation[2] provides a description of an isolated loop of helical motion along the vortex line. In obtaining these results, the Serret-Frenet equations play an auxiliary role with respect to the nonlinear Schrödinger equation that is analogous to the one played by the Bloch equations in determining soliton behavior in coherent optical pulse propagation.[3] The Bloch equations, of course, have the structure of a single component of the vector Serret-Frenet equations. The many similarities in the properties of the various soliton equations tempt one to conjecture that other equations exhibiting soliton behavior may also be related to helical spece curves and that such geometric considerations may play a fundamental role in understanding soliton propagation. Furthermore, an association of nonlinear evolution equations with geometric concepts of a somewhat more abstract nature has already appeared.[4] One's expectations in this matter are also enhanced by the recognition that equations of Serret-Frenet type lead directly[3] to linear equations of the form used to solve the various soliton equations by the two-component inverse method.[2,5] Geometric interpretation of these equations could thus provide a natural motivation for the introduction of the inverse-scattering approach that would be analogous to that already provided by the Bloch equations in coherent optical pulse propagation.

It has been found that the so-called sine-Gordon equation may be associated with curves of constant curvature and that this curvature is related to the eigenvalue parameter in the inverse-scattering formulism. The modified Korteweg–de Vries equation, on the other hand, may be associated with curves of constant torsion, the torsion playing the role of eigenvalue parameter. For the nonlinear Schrödinger equation, both curvature and torsion may vary, and it is the asymptotic value assumed by the torsion at large distances from the disturbance that is the eigenvalue parameter.

The spatial variations of a twisted curve are governed by the Serret-Frenet equations

$$\hat{t}_s = \kappa \hat{n}, \tag{1a}$$

$$\hat{b}_s = -\tau \hat{n}, \tag{1b}$$

$$\hat{n}_s = \tau \hat{b} - \kappa \hat{t}, \tag{1c}$$

where the subscript denotes partial differentiation with respect to an arc length parameter s. The functions $\tau(s,t)$ and $\kappa(s,t)$ are the torsion and curvature, respectively, while $\hat{t}$, $\hat{n}$, and $\hat{b}$ are the usual tangent, normal, and binormal to the curve. Also, $\hat{t}=\vec{X}_s(s,t)$, where $\vec{X}(s,t)$ is the position vector to a point s on the curve at time t. If it is assumed that the torsion τ approaches a constant value τ_0 in regions of the curve that are remote from the disturbance in question, then Eqs. (1) may be combined to yield

$$\vec{N}_s + i\tau_0\vec{N} = -\psi\hat{t}, \tag{2a}$$

$$\hat{t} = \tfrac{1}{2}(\psi^*\vec{N} + \psi\vec{N}^*), \tag{2b}$$

where

$$\vec{N} \equiv (\hat{n}+i\hat{b})\exp[i\textstyle\int_{-\infty}^{s} ds'(\tau-\tau_0)], \tag{3a}$$

$$\psi \equiv \kappa\exp[i\textstyle\int_{-\infty}^{s} ds'(\tau-\tau_0)]. \tag{3b}$$

Following Hasimoto, one may express the temporal variation of $\vec{N}$ and $\hat{t}$ as linear combinations of $\vec{N}$, $\vec{N}^*$, and $\hat{t}$, i.e.,

$$\vec{N}_t = \alpha\vec{N} + \beta\vec{N}^* + \gamma\hat{t}, \tag{4a}$$

$$\hat{t}_t = \lambda\vec{N} + \mu\vec{N}^* + \nu\hat{t}. \tag{4b}$$

For the position vector, one may write $\vec{X}_t = \kappa(\xi\hat{n} + \eta\hat{b}) + \theta\hat{t}$ or, equivalently,

$$\vec{X}_t = C^*\psi^*\vec{N} + C\psi\vec{N}^* + \theta\hat{t}, \quad C = \tfrac{1}{2}(\xi + i\eta). \tag{5}$$

Multiplication of Eqs. (4) by $\vec{N}$ and $\hat{t}$ and use of the relations $\vec{N}\cdot\vec{N}^*=2$, $\vec{N}\cdot\hat{t}=\vec{N}\cdot\vec{N}=0$ yields $\alpha+\alpha^*=0$, $\beta=\nu=0$, and $\gamma=-2\mu$. Hence

$$\vec{N}_t = iR\vec{N} + \gamma\hat{t}, \tag{6a}$$

$$\hat{t}_t = -\tfrac{1}{2}(\gamma^*\vec{N} + \gamma\vec{N}^*), \tag{6b}$$

where $R(s,t)$ is real. Equating mixed second derivatives from Eqs. (2) and (6) one finds[1]

$$\psi_t + \gamma_s + i(\tau_0\gamma - R\psi) = 0, \tag{7a}$$

$$R_s = \tfrac{1}{2}i(\gamma\psi^* - \gamma^*\psi). \tag{7b}$$

Similarly, the requirement $\vec{X}_{st} = \vec{X}_{ts}$ yields

$$-\tfrac{1}{2}\gamma = (C\psi)_s + i\tau_0 C\psi + \tfrac{1}{2}\theta\psi, \tag{8a}$$

$$\theta_s = \xi|\psi|^2. \tag{8b}$$

Combination of Eqs. (7b) and (8a) gives

$$R_s = \eta_s|\psi|^2 - \tfrac{1}{2}i\xi(\psi^*\psi_s - \psi\psi_s^*) + \tfrac{1}{2}\eta|\psi|_s^2 + i\tau_0\theta_s. \tag{9}$$

Equations (7) and (8) provide six equations for the eight functions contained in R and θ and the complex terms ψ, γ, and C. This indeterminacy may be used to specialize certain of these quantities to yield simple space curves, as well as to simplify the analysis. Equation (7a) can then become one of the standard nonlinear evolution equations. At the same time Eqs. (2) and (6) reduce to the linear equations solved by the two-component inverse-scattering method.

Reduction to the inverse-scattering equations follows upon the recognition that any of the three *scalar* components of Eqs. (2) and (6) possess the first integral $N^2+t^2=1$, where N is now some one component of $\vec{N}$ and similarly for t. Setting $N=u+iv$ and following a standard procedure[6,3] one may factor this first integral and write

$$\frac{u+it}{1-v} = \frac{1+v}{u-it} \equiv \varphi, \tag{10a}$$

$$\frac{u-it}{1-v} = \frac{1+v}{u+it} \equiv -\frac{1}{\chi} = \varphi^*. \tag{10b}$$

When this one component of the Serret-Frenet equations (analogous to the Bloch equations) is expressed in terms of φ and χ one obtains the Riccati equation

$$\varphi_t + i\gamma_r\varphi + \tfrac{1}{2}(i\gamma_i - R)\varphi^2 - \tfrac{1}{2}(i\gamma_i + R) = 0, \tag{11}$$

where $\gamma = \gamma_r + i\gamma_i$. An identical equation obtains for χ. If this Riccati equation is replaced by a pair of first-order linear equations instead of the usual second-order equation, one finds

$$v_{1t} = \tfrac{1}{2}i\gamma_r v_1 + \tfrac{1}{2}(i\gamma_i - R)v_2, \tag{12a}$$

$$v_{2t} = (i\gamma_i + R)v_1 - \tfrac{1}{2}i\gamma_r v_2, \tag{12b}$$

where $\varphi = v_2/v_1$.

In like manner, the spatial dependence of Eq. (2) is equivalent to

$$v_{1s} = -\tfrac{1}{2}i\psi_r v_1 + \tfrac{1}{2}(-i\psi_i + \tau_0)v_2, \tag{13a}$$

$$v_{2s} = -\tfrac{1}{2}(i\psi_i + \tau_0)v_1 + \tfrac{1}{2}i\psi_r v_2. \tag{13b}$$

In a similar way, the alternative factorization $(u+iv)/(1-t) = (1+t)/(u-iv) = \varphi$, etc., leads to the pairs

$$v_{1t} = -\tfrac{1}{2}iRv_1 + \tfrac{1}{2}i\gamma^* v_2, \tag{14a}$$

$$v_{2t} = -\tfrac{1}{2}i\gamma v_1 + \tfrac{1}{2}iRv_2, \tag{14b}$$

and

$$v_{1s} - \tfrac{1}{2}i\tau_0 v_1 = -\tfrac{1}{2}\psi^* v_2, \tag{15a}$$

$$v_{2s} + \tfrac{1}{2}i\tau_0 v_2 = \tfrac{1}{2}\psi v_1. \tag{15b}$$

The factorization $(v+it)/(1-u)$ etc., could also be considered but will not be employed here. When the functions γ and R have been chosen to reduce Eq. (7a) to a specific evolution equation, either Eqs. (12) and (13) or (14) and (15) immediately provide the familar linear systems that are customarily associated with these equations in an *ad hoc* manner for solution by the inverse method.

(i) *Curves of constant curvature: sine-Gordon equation.*—In this instance $\psi = \kappa_0 \exp(i\sigma)$, where κ_0 is the constant value of the curvature and $\sigma_s = \tau$. (Considerations are further specialized to the case $\tau_0 = 0$.) If one sets $\gamma_i = 0$, then $R = \sigma_t$ according to the imaginary part of Eq. (7a). From the real part of Eq. (7a), $\gamma_r = f(t)$ an arbitrary function of time. Then Eq. (7b) yields $R_s = \kappa_0 f(t) \times \sin\sigma$. Introducing the new time coordinate $dt' = \kappa_0 f(t)\, dt$ one obtains

$$\sigma_{st'} = \sin\sigma, \tag{16}$$

More simply, one may choose $\gamma_r = \text{const}\ (=\gamma_{r0})$ and then set $\gamma_{r0}\kappa_0 = 1$. With these values for γ, R, and ψ, Eqs. (12) and (13) become the equations used to solve Eq. (16) by the inverse method with the eigenvalue parameter equal to $1/2\kappa_0$.

(ii) *Curves of constant torsion: modified Korteweg–de Vries equation.*—For curves of constant torsion ($=\tau_0$), ψ in Eq. (3b) is seen to be real. Equations (7a) and (8a) are readily combined to yield an equation for ψ involving third spatial derivatives. The simplest assumption of $\eta = $ constant is found to be inadequate for nonsteady-state propagation. Setting $\eta = \tau_0 + \rho$ and then exploiting the above-mentioned indeterminacy to impose the requirement $(\rho\psi)_s + \tau_0 \zeta \psi = 0$ (since this expression arises repeatedly in the analysis), one obtains the equation $\psi_t + \frac{3}{2}\psi^2\psi_s + \psi_{sss} = 0$ with $\gamma_r = \frac{1}{2}\psi^3 + \psi_{ss} + \tau_0^2\psi$, $\gamma_i = -\tau_0\psi_s$, and $R = \frac{1}{2}\tau_0\psi^2 + \tau_0^3$. With $\psi = 2u$, these expressions in conjunction with Eqs. (14) and (15) become the standard results for the modified Korteweg–de Vries equation.

(iii) *An integrable expression for R_s: nonlinear Schrödinger equation.*—As a simple example in which neither curvature nor torsion is assumed to be constant, one may ask for a choice of parameters that permits integration of the expression for R given in Eqs. (7b) or (9). An obvious choice is $\zeta = 0$, $\eta = \text{const} = \eta_0$. Then $R = \frac{1}{2}\eta_0 |\psi|^2 + \Gamma(t)$, where $\Gamma(t)$ arises from integration. Also, $\gamma = i\eta_0\psi_s + (\tau_0\eta_0 + \theta)\psi$. A standard form for the nonlinear Schrödinger equation follows by setting $\eta_0 = 2$ and choosing $\theta(t) = 4\tau_0$ and $\Gamma(t) = -2\tau_0^2$. Then $R = |\psi|^2 - 2\tau_0^2$ and $\gamma = -2i\psi_s - 2\tau_0\psi$ with ψ satisfying

$$i\psi_t + 2\psi_{ss} + |\psi|^2\psi = 0. \tag{17}$$

Equations (14) and (15) yield the usual linear equations associated with the nonlinear Schrödinger equation.

The motion of helical curves, along with providing a three-dimensional context for soliton propagation, leads quite naturally to the systems of linear equations that are solved by the inverse method. Consideration of solitons in the context of torsional waves also provides motivation for associating vanishing reflection coefficient with soliton propagation. This result is analogous to that found previously in coherent optical pulse propagation.[3] Such topics will be considered subsequently.

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