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Exact Self-Consistent Equilibria of Relativistic Electron Beams*

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Fully self-consistent and relativistically exact solutions for the equilibrium of a nonneutral, cold, relativistic, electron beam in a magnetized conducting pipe reveal that the beam is neither of uniform density nor a rigid rotor when total momenta and energy are uniform. The profiles are simple rational functions of the radius. Voltage-current-magnetization characteristics are obtained, showing a maximum current and a cutoff in magnetization for fixed injection voltage.

Equilibria of cold, nonneutral, relativistic, electron beams[1] have often been derived from a force-balance equation that represents one constraint on three distributions, typically those of density, rotation, and axial velocity. Equilibrium and stability analyses usually assume that the beam has a uniform density profile or rotates rigidly.[2] Such assumptions lead to equilibria constrained to have specific, nonuniform distributions of momenta and energy. The fully self-consistent, relativistically exact calculation below instead prescribes that all electrons are to have the same total (mechanical plus electromagnetic) momentum and energy. This is found to allow neither a rigid rotor nor a uniform density of the beam.[3]

An infinitely long, nonneutral, cold, relativistic electron beam propagates inside a circular conducting pipe, confined by an externally imposed axial magnetic field, B_0. The evacuated pipe has radius a, while the coaxial beam extends to radius r_0 where it has a sharp edge. The beam streams axially with mean velocity $v=\beta c$ and rotates about the axis, driven by the combined electric and magnetic fields. The fields expected in this system include a radial electric field $E_r(r)$ due to the nonneutralized charge, an azimuthal magnetic field $B_\varphi(r)$ from the streaming current, and an internally generated axial magnetic field $B_z(r)$ due to the rotation of the beam. These are derivable from an axisymmetric potential that has both axial and azimuthal components.

Space-time points are identified by a position four-vector x_α with imaginary temporal component $x_4=ict$. The mean velocity four-vector of the beam, u_α, has nonzero components $u_3=\gamma v$ and $u_4=ic\gamma$, where $\gamma=(1-\beta^2)^{-1/2}$. The cross product with the axial unit vector is expressed by the antisymmetric tensor $Z_{\alpha\beta}$ whose nonzero components are $Z_{12}=-Z_{21}=-1$. The four-potential, $A_\alpha(x)$, that corresponds to the expected field pattern is

$$(e/m)A_\alpha(x)=p(r)u_\alpha+q(r)(\omega_c/2)Z_{\alpha\beta}x_\beta, \qquad (1)$$

where $(-e/m)$ is the ratio of electron charge to rest mass and $\omega_c=(e/m)B_0$ is the cyclotron frequency in the applied magnetic field. The two unknown radial profiles, $p(r)$ and $q(r)$, are to be determined self-consistently. In the absence of the beam, there is only the applied uniform magnetization, expressed by $p=0$ and $q=1$.

The four-potential satisfies the boundary conditions at the surface of the conducting pipe for any profiles p, q. It also satisfies the Lorentz gauge condition $\partial A_\alpha/\partial x_\alpha=0$. The inhomogeneous wave equation is hence

$$\partial^2 A_\alpha/\partial x_\beta \partial x_\beta=-\mu_0 J_\alpha, \qquad (2)$$

which yields the four-current density, $J_\alpha(x)$, that accompanies and generates the potential,

$$-\mu_0\frac{e}{m}J_\alpha=\frac{1}{r}\frac{d}{dr}\left(r\frac{dp}{dr}\right)u_\alpha+\frac{1}{r^3}\frac{d}{dr}\left(r^3\frac{dq}{dr}\right)\frac{\omega_c}{2}Z_{\alpha\beta}x_\beta. \qquad (3)$$

This has the same form as the potential and satisfies the continuity equation $\partial J_\alpha/\partial x_\alpha = 0$ for any p, q. It displays axial streaming and azimuthal rotation and its temporal component gives the charge-density distribution.

The current density is provided by the beam itself and can hence be expressed alternatively through the flow field $v_\alpha(x)$ and the invariant particle density $n(x)$, as $J_\alpha = -nev_\alpha$, or

$$-\mu_0(e/m)J_\alpha(x) = [\omega_p^2(r)/c^2]v_\alpha(x), \quad (4)$$

where $\omega_p(r)$ is the plasma frequency corresponding to the invariant density. As is true for any four-velocity, the flow field must satisfy the constraint

$$v_\alpha(x)v_\alpha(x) = -c^2, \quad (5)$$

to conform with the principle of relativity.

A typical beam electron under the influence of the fields in the system, represented covariantly by the electromagnetic field four-tensor $B_{\alpha\beta} = \partial A_\alpha/\partial x_\beta - \partial A_\beta/\partial x_\alpha$, follows trajectory $x_\alpha(\tau)$ where τ is the proper time along the world line. The equation of motion prescribed by the relativistic Lorentz force law is

$$d^2x_\alpha/d\tau^2 = (e/m)B_{\alpha\beta}(x)(dx_\beta/d\tau), \quad (6)$$

in which the field is evaluated at the instantaneous location of the electron, $x = x(\tau)$.

The motion of each electron in the beam's self-consistent fields must conform exactly to the flow of the beam at the location of the electron:

$$dx_\alpha/d\tau = v_\alpha(x), \quad (7)$$

where the flow field is evaluated at $x = x(\tau)$. This is the cold-beam condition; for a warm beam, the relation holds only between the averaged quantities. Under this equilibrium condition, the acceleration becomes a convective derivative of the flow field, $(\partial v_\alpha/\partial x_\beta)v_\beta$, and (6) becomes

$$\frac{\partial v_\alpha}{\partial x_\beta}v_\beta = \frac{\partial[(e/m)A_\alpha]}{\partial x_\beta}v_\beta - \frac{\partial[(e/m)A_\beta]}{\partial x_\alpha}v_\beta. \quad (8)$$

In view of (5), a solution for $v_\alpha(x)$ is evidently given by the potential field within the beam,

$$v_\alpha(x) = (e/m)A_\alpha(x). \quad (9)$$

This is also the condition for the vanishing of the canonical four-momentum of the system for this equilibrium.

This solution allows the flow field to be expressed directly as

$$v_\alpha(x) = p(r)u_\alpha + \tfrac{1}{2}q(r)\omega_c Z_{\alpha\beta}x_\beta. \quad (10)$$

The two unknown profiles $p(r)$, $q(r)$ are now not independent; they are related by the constraint (5) on the flow velocity, $v_\alpha v_\alpha = -c^2$, which becomes

$$p^2(r) - (\omega_c^2 r^2/4c^2)q^2(r) = 1, \quad (11)$$

within the beam. Additional equations relating the profiles are now obtained by equating the two forms for the four-current density in (3) and in (4) with (10). This yields two differential equations for p, q, involving the density profile $\omega_p^2(r)$ as a third unknown function:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dp}{dr}\right) = \frac{\omega_p^2(r)}{c^2}p(r), \quad (12)$$

$$\frac{1}{r^3}\frac{d}{dr}\left(r^3\frac{dq}{dr}\right) = \frac{\omega_p^2(r)}{c^2}q(r). \quad (13)$$

Equations (11)–(13) are three simultaneous, nonlinear differential and algebraic equations for the three unknown profiles $p(r)$, $q(r)$, and $\omega_p(r)$. They determine all the equilibrium properties self-consistently and allow no arbitrary selections of density or rotation profiles. The Bessel-function solutions[4] to (12) and (13) when a uniform density is presupposed are not compatible with (11). Similarly, a presumption of a rigid rotor is also inconsistent with the three equations.

The profiles that solve the three equations are simple rational functions of the radius. The solutions become singular at a certain radius, R, a parameter or integration constant of the equations. The singularity cannot, of course, appear within the beam. Inside the beam, for $r \le r_0$, the solutions are

$$p(r) = \frac{R^2 + r^2}{R^2 - r^2}, \quad (14)$$

$$q(r) = \frac{4c}{\omega_c R}\frac{R^2}{R^2 - r^2}, \quad (15)$$

$$\omega_p^2(r) = \frac{8c^2}{R^2}\frac{R^4}{(R^2 - r^2)^2}. \quad (16)$$

Between the beam and the pipe, the density is zero and $p(r)$ and $q(r)$ satisfy the linear equations obtained by setting $\omega_p(r) = 0$ in (12) and (13). They are matched to the internal solutions by imposing continuity of the fields at the boundary $r = r_0$. The external solutions, for $r_0 \le r \le a$, are $\omega_p^2(r) = 0$ and

$$p(r) = \frac{R^4 - r_0^4 + 4R^2 r_0^2 \ln(r/r_0)}{(R^2 - r_0^2)^2}, \quad (17)$$

$$q(r) = Q(1 - r_0^4/R^2 r^2), \quad (18)$$

where $Q=(4cR^3/\omega_c)/(R^2-r_0^2)^2$.

The parameter R, which must exceed the beam radius r_0, is determined by known system properties, such as the applied magnetization B_0. The axial magnetic field is the B_{21} component of the electromagnetic field tensor,

$$B_{21}=\frac{m}{e}\omega_c\left(q+\frac{r}{2}\frac{dq}{dr}\right). \tag{19}$$

Outside the beam, this is only the applied magnetization, B_0, which requires $Q=1$ in (18). R is hence obtainable from the applied magnetic field through

$$\omega_c r_0/4_c=r_0/R[1-(r_0/R)^2]^2. \tag{20}$$

The accelerating voltage applied between cathode and pipe is $(\gamma_I-1)mc^2/e$, in terms of the injection energy. A beam electron traveling along the axis has been accelerated by voltage $(\gamma-1)mc^2/e$. The temporal component of the four-potential gives the additional potential difference between pipe and beam axis as $\gamma[p(a)-p(0)]mc^2/e$. The applied voltage is hence related to γ and to R by

$$\gamma_I=\gamma p(a), \tag{21}$$

where $p(a)$ is to be read from (17).

Integration of the axial and temporal components of the four-current density across the area of the beam yields, respectively, the total beam current, I, and the Budker parameter, ν. In terms of the Alfvén current $I_A=I_{17}\beta\gamma$, where $I_{17}=4\pi\epsilon_0 mc^3/e=17$ kA,

$$\frac{I}{I_A}=\frac{\nu}{\gamma}=\frac{r_0}{2}\frac{dp(r_0)}{dr}=\frac{2(r_0/R)^2}{[1-(r_0/R)^2]^2}. \tag{22}$$

The transverse electromagnetic field profiles are given by dp/dr. The axial component of the Poynting vector integrated over the cross section of the pipe gives the total power carried by the fields. The ratio of this to the square of the current is the equivalent impedance:

$$Z_0=\frac{\eta_0}{2\pi}\frac{\ln(a/r_0)}{\beta}+\frac{\eta_0}{8\pi\beta}\left(1-\frac{r_0^2}{R^2}\right)\left(1-\frac{r_0^2}{3R^2}\right), \tag{23}$$

where $\eta_0=(\mu_0/\epsilon_0)^{1/2}=377\ \Omega$. The two terms are the impedance of the coaxial beam-pipe configuration and the contribution of the beam region.

The beam motion is obtained by integrating (7) using (10). Since $dx_\alpha/d\tau$ has no radial component, the radius r is a constant of the motion and (7) integrates to

$$x_\alpha(\tau)=pu_\alpha\tau+\exp(\tfrac{1}{2}\omega_c\tau qZ_{\alpha\beta})x_\beta(0). \tag{24}$$

The first term is an axial drift and the second, expressible in terms of trigonometric functions, represents circular rotation. In the laboratory frame, this is helical motion about the axis with pitch $l(r)=4\pi(c/\omega_c)\gamma\beta p(r)/q(r)$ and rotation rate

$$\omega(r)=\frac{d\varphi/d\tau}{dt/d\tau}=\frac{\omega_c}{2\gamma}\frac{q(r)}{p(r)}=\frac{2cR/\gamma}{R^2+r^2}. \tag{25}$$

There is shear in the rotation velocity; the beam is not a rigid rotor.

The beam is also not uniform in density. As given by the temporal component of $J_\alpha(x)$, the particle density in the laboratory frame is

$$n_0(r)=\frac{m\epsilon_0}{e^2}\gamma\omega_p^2(r)p(r)=\frac{m\gamma}{e^2\mu_0}\frac{8R^2(R^2+r^2)}{(R^2-r^2)^3}. \tag{26}$$

The electrons are evidently massed toward the edge of the beam.

Figure 1 presents relative profiles of density, axial magnetic field, and rotation rate for a beam with parameter $R=2r_0$. This choice corresponds to a beam strength $\nu/\gamma=0.89$, an injection energy $\gamma_I=1.1\gamma$, and a confining magnetic field given by $\omega_c r_0/c=3.56$. The density varies steeply, peaking at the edge at 3 times the axial density. The rotation rate drops by 20% from axis to edge. The axial magnetic field is B_0 outside the beam but is depressed diamagnetically inside to somewhat more than half the applied magnetization.

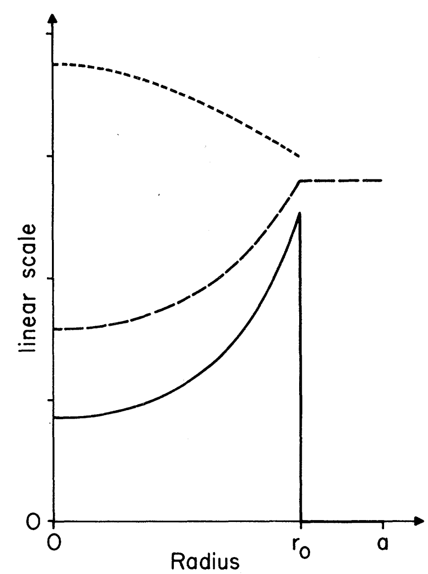

FIG. 1. Profiles of laboratory-frame density $n_0(r)$ (solid curve), axial magnetic field $B_z(r)$ (dashed curve), and rotation rate $\omega(r)$ (dotted curve) for a beam of radius r_0 in a pipe of radius $a=\frac{4}{3}r_0$, with parameter $R=2r_0$.

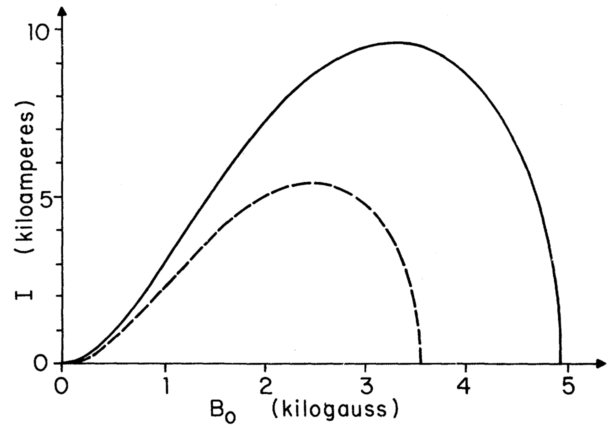

FIG. 2. Total beam current I versus applied magnetization B_0 for fixed injection voltages, $\gamma_I = 2.5$ (solid curve) and $\gamma_I = 2.0$ (dashed curve). Beam radius $r_0 = 1.5$ cm, pipe radius $a = 2$ cm.

Combining the relations of the voltage, current, and magnetization to parameter R gives the electrical-circuit properties of the beam. Figure 2 shows the total beam current as a function of applied magnetization, for two values of applied voltage and for fixed geometry. Weak magnetic fields can confine only a weak beam. For strong magnetization, most of the energy appears as transverse rotational motion, rather than streaming, and the current is again low. The current reaches a peak at some optimal magnetization. There is an abrupt cutoff at a maximal magnetization at which all the applied energy is converted to circular motion. Beyond this, the equilibrium state described here can no longer exist. The maximum current attainable at a given voltage exceeds the usual predicted value[5] not only because the density is not uniform but also because its profile varies with magnetization.

The equilibrium satisfies the cold-fluid radial force-balance equation[1] but also treats the constituent electrons equally in requiring all to have the same canonical linear momentum, angular momentum, and energy, including both the mechanical and field contributions. The axial velocity is found to be uniform, since the axial and temporal flow field components share a common profile. Allowance for different profiles merely leads to four equations in four unknowns, instead of three, and the only solution to be found then is identical to the one presented herein. The solution can readily be extended to apply to a hollow, annular beam,[6] with characteristics similar to those of the full beam, but with both slow and fast rotations permitted.

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