COMMENTS

Lowering of Dimensionality in Phase Transitions with Random Fields*

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We prove that to all orders in perturbation expansion, the critical exponents of a phase transition in a d-dimensional ($4<d<6$) system with short-range exchange and a random quenched field are the same as those of a $(d-2)$-dimensional pure system. Heuristic arguments are given to discuss both this result and the random-field Ising model for $2<d<6$.

The effects of a random quenched magnetic field (or, in general, field conjugate to the order parameter) on a magnetic (other) phase transition have recently been studied by several authors.[1-3] Renormalization-group arguments show[1] that for such systems the classical mean-field-like behavior is found at dimensions d above $d_c=6$, instead of at $d_c=4$ for the "pure" systems. Similarly, for systems with continuous symmetry ($n\geq 2$, where n is the number of spin components), these systems have no long-range order for $d<d_d=4$ (compared to $d_d=2$).[1] For Ising systems ($n=1$), $d_d=2$ (compared to $d_d=1$).

The results for $n\geq 2$, together with a comparison of the renormalization-group calculation of critical exponents to order ϵ ($\epsilon=6-d$)[1] with their counterparts in the usual ϵ expansion for pure systems (where $\epsilon=4-d$), seem to indicate that *critical properties of continuously symmetric systems with random fields in d dimensions are closely related to those of pure systems in d−2 dimensions.* Indeed, Grinstein[2] observed that the hyperscaling law $d\nu=2-\alpha$ (ν and α are the correlation-length and specific-heat exponents) is violated for the random-field systems, being replaced by $(d-2)\nu=2-\alpha$ up to order ϵ^3. Similar results were earlier found exactly for the ideal Bose gas, which is equivalent to the limit $n=\infty$.[3]

In this note we prove exactly, that *the d-dimensional random-field problem is indeed equivalent to the (d−2) dimensional pure problem, term by term, in perturbation theory.* In particular, this implies that the ϵ expansions ($d=6-\epsilon$ for "random field" and $d=4-\epsilon$ for "pure") of all critical exponents are the same to all orders. Similarly, the $1/n$ expansions (for $4<d<6$, random; and $2<d<4$, pure) of these quantities are identical to all orders. Within these expansions, all random hyperscaling laws at d dimensions must have $d-2$ instead of d in the appropriate formulas.

We interpret this result physically by noting that sufficiently close to T_c *the dominant disordering agent becomes the random field instead of the thermal fluctuations.* A heuristic argument is used to show that the former effective disordering energy for a coherence volume ξ^d is now proportional to ξ^2, instead of k_BT ($\sim\xi^0$). This reduces the effective dimensionality of the problem by two.

Since for $n=1$, d_d is equal to 2 (instead of $d_d=1+2=3$), it is clear that the rule $d\to d-2$ may not apply for $n=1$ at low dimensionalities. This is probably due to the nonconvergence of the expansions for large ϵ and $1/n$. We shall therefore give a separate, heuristic, discussion of the Ising random case at the end.

Following Ref. 1, we start with the Ginzburg-Landau-Wilson (GLW) Hamiltonian

$$\mathcal{H}=-\int d^dx\{\tfrac{1}{2}[r|\vec{\sigma}(\vec{x})|^2+(\nabla\vec{\sigma})^2]+u|\vec{\sigma}|^4+\vec{h}(\vec{x})\cdot\vec{\sigma}(\vec{x})\},\tag{1}$$

where $\vec{\sigma}(\vec{x})$ is the order parameter, and $\vec{h}(\vec{x})$ is a random-field variable, with $\bar{h}=0$ and (Gaussian) correlations (in Fourier space)

$$\overline{h^\alpha(\vec{k})h^\beta(\vec{k}')}=\delta_{\alpha\beta}\delta(\vec{k}+\vec{k}')f(\vec{k}),\tag{2}$$

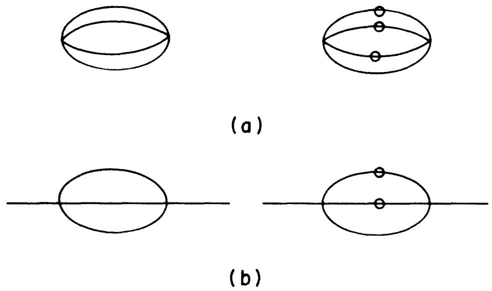

FIG. 1. Typical diagrams which appear in the Feynman graph expansion. Vertices denote the GLW four-spin coupling u, while empty circles denote the random field variable λ. (a) Diagrams for the free energy; (b) diagrams for the spin-spin correlation function.

where $f(0)=\lambda \neq 0$, and the bar denotes averaging over random-field configurations.

For a given (quenched) distribution of magnetic fields, we expand all thermodynamic functions in a perturbation expansion in both u and $\vec{h}(\vec{k})$, and then average over all random-field configurations. Following Wilson,[4] we now study the most divergent terms in this expansion, and fit them to scaling predictions. It turns out[1,2] that the small expansion variable is now $u\lambda$ (instead of u). For the ϵ expansion ($\epsilon = 6-d$) this is fixed at a critical value $(u\lambda)_c$, so that scaling is obeyed, by considering the four-point correlation function.[4] Since we use a diagramatic expansion, and not renormalization-group recursion relations, we do not have to worry about higher-order vertices or other irrelevant variables.[4]

Diagrammatically, each four-spin interaction is represented by a vertex, while each factor $\vec{h}(\vec{k})$ is represented by an "external" line in the diagram, ending at a point which has the variable $\vec{h}(\vec{k})$ attached to it. Upon averaging, the only remaining nonvanishing diagrams will be those in which pairs of these "field" lines meet, each creating an internal line with *two* propagators and with an "empty circle," denoting the variable λ, on its middle. Figure 1 shows some examples of diagrams which will appear in the perturbation expansions of the free energy and of the spin-spin correlation function. The most divergent

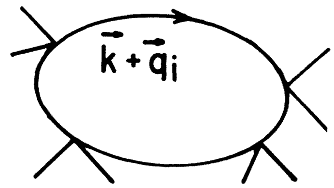

FIG. 2. A typical one-loop diagram, with $l=4$ internal lines.

diagrams will now be those which arise from *treelike* diagrams, on which each external line involves the field variable $\vec{h}(\vec{k})$. On averaging over the field configurations, pairs of field lines will meet, and the resulting diagrams will be similar to those of the nonrandom problem, except that now *each loop will have exactly one "circled" internal line* (see Fig. 1). Any nontreelike diagram will have loops with no circled lines, thus will involve less propagators, and therefore will be less divergent. Loops with more circled lines arise from disconnected diagrams, which should not be included.

We next compare the results for pure diagrams with those for circled ones. The latter involve the effects of the *random field,* while the former involve those of *thermal disorder,* or entropy. Since the circled diagrams are more divergent, one concludes that in the present problem thermal disorder is much less relevant than in the pure problem. We shall discuss this further below. Our basic result is that *each circled loop integral in d is simply to be replaced by its pure counterpart in $d-2$ dimensions, apart from a multiplicative factor* $(4\pi\lambda)$. This can be absorbed in the appropriate volume factor K_{d-2},[4] and thus will not affect critical exponents. It may, however, affect universal amplitude combinations which depend on K_{d-2}.[5] Since the free energy is also given by a diagrammatic expansion [e.g., Fig. 1(a)], the power $d\nu$ which appears in it will also be replaced by $(d-2)\nu$.

The main part of the proof concerns a single loop, with l internal lines (Fig. 2). Without the random field, there are no empty circles on any of the internal lines, and the loop integral is given by[6]

$$I_l^{\,d}(\vec{q}_i, r_i) = (2\pi)^{-d}\int d^dk \prod_{j=i}^{l}[r_j + (\vec{k}+\vec{q}_j)^2]^{-1}$$
$$= (l-1)!(2\pi)^{-d}\int d^dk \int_0^1 d\alpha_1 \ldots \int_0^1 d\alpha_l\, \delta\Big(\sum_{j=1}^{l}\alpha_j - 1\Big)\Big[\vec{k}^2 + \sum_{j=1}^{l}\alpha_j(r_j+\vec{q}_j^{\,2})\Big]^{-l}. \quad (3)$$

We have exploited the arbitrariness in choosing the momenta of the lines as $(\vec{k}+\vec{q}_j)$, so that $\sum_j \alpha_j\vec{q}_j = 0$.[6] To obtain the contribution of the same loop to the random-field problem, we have to circle one of the internal lines, multiply by λ, and sum over all possible circlings. This yields

$$\tilde{I}_l{}^d(\vec{q}_i, r_i) = -\sum_{j=1}^{l} \lambda \frac{\partial I_l{}^d}{\partial r_j} = \lambda l!(2\pi)^{-d}\int d^dk \int_0^1 d\alpha_1 \ldots \int_0^1 d\alpha_l\, \delta(\sum_j \alpha_j - 1)[\vec{k}^2 + \sum_j \alpha_j(r_j + \vec{q}_j{}^2)]^{-l-1}. \tag{4}$$

We now divide the momentum $\vec{k}$ into two parts, $\vec{k}_1$ with $d-2$ components and $\vec{k}_2$ with two components. Integration over $\vec{k}_2$ and comparison with (3) immediately yield the result

$$\tilde{I}_l{}^d(\vec{q}_i, r_i) \equiv 4\pi\lambda I_l{}^{d-2}(\vec{q}_i, r_i). \tag{5}$$

Consider now a diagram with many loops. Having proved (5) for one loop, we next consider another loop. If the two loops have common lines, then we consider only those configurations of the circled diagrams for which these lines were not circled during the calculation of the first loop integral. For these we now repeat the calculation. In intermediate steps, the momenta $\vec{q}_j$ have d components, and not $d-2$ components, as seems to be required by the right-hand side of (5). However, $I_l{}^{d-2}$ is actually a function only of the combination $(r_j + \vec{q}_j{}^2)$, where $\vec{q}_j{}^2$ may be treated as a numerical parameter. After finishing all the loop integrals, the result will be a function of the external momenta only. As long as there are no more than $d-2$ of such momenta, we can always choose them to be in the $(d-2)$-dimensional subspace on which $I_l{}^{d-2}$ is defined. In practice, for $d>4$, this will be sufficient for calculating all the necessary exponents.

There are various possible generalizations of the present study. For one, one may consider long-range correlations in the random field, so that $f(\vec{k}) \propto k^\theta$ for $\vec{k} \to 0$ in Eq. (2).[3] This will introduce an additional power of $|\vec{k} + \vec{q}_j|^\theta$ into the integrals in Eq. (4), and thus lower the values of d_c and d_a, presumably by θ. It is not clear yet if this shift in dimensionality will apply to all exponents, to all orders.

The physical reason for the effective lowering of d has to do with the fact that sufficiently close to T_c the random field becomes the dominant cause for disorder. In fact, the average of the square of the energy due to the random field, for a coherence volume ξ^d, is

$$\overline{\langle \Delta E_h \rangle^2} \propto \lambda \xi^d \overline{\langle \vec{\sigma}(0) \rangle^2}. \tag{6}$$

Here the angular brackets denote the statistical average at a given random-field configuration. If this is used in the Ginzburg[7] criterion for the critical region, one easily finds that the dimension where classical behavior is valid is indeed $d>6$, and that the effective coherence volume is ξ^{d-2}.

In order to understand qualitatively the effect of this disordering energy on the critical behavior and the hyperscaling laws,[2] one may use the Pippard demonstration of hyperscaling, as quoted by Kadanoff.[8] It is assumed that the characteristic fluctuation energy of a coherence volume, $t^{2-\alpha}\xi^d$, is of the same order of magnitude as $[\overline{\langle \Delta E_h \rangle^2}]^{1/2}$. To evaluate the latter quantity, one may note that at each coherence domain the spins are slightly disturbed by the excess random field, which is of order $\xi^{-d/2}$. Thus one may write [for a typical value of $\langle\sigma(0)\rangle$] $\langle\sigma(0)\rangle \sim t^{-\gamma}\xi^{-d/2}$ and thus $[\overline{\langle \Delta E_h \rangle^2}]^{1/2} \sim \lambda t^{-\gamma}$. This result can also be obtained by summing the energy for each $\vec{k}$ component, $\chi(\vec{k}) \times [\overline{h(\vec{k})^2}]$, over $|k| < 1/\xi$ and dividing by the number of coherence volumes in the system, $V\xi^{-d}$. If the small critical index η is now neglected in this heuristic argument, one obtains $t^{2-\alpha}\xi^d \sim \lambda\xi^2$, i.e., $2-\alpha = \nu(d-2)$. Note again that the physical reason for the change in the "effective dimensionality" is that the disordering energy for a coherence volume is not of $O(1)$, but proportional to a positive power (presumably 2) of the coherence length! We have no explanation for the fact that η must be neglected in order to reproduce the diagrammatic result. Moreover, it is interesting to note that if η is kept, then d is replaced by $d-2+\eta$, instead of $d-2$. For $n=1$, $d=1$, one has $\eta=1$. Could this explain the shift in d_a by 1, instead of 2, for the Ising case? We leave these intriguing questions for future study.

To obtain results for $n=1$, $d=3$, we must use the ϵ expansion with $\epsilon=3$, or the $1/n$ expansion with $1/n=1$. These values are very probably out of the radii of convergence of the appropriate series. The question of the violation of hyperscaling for this case[2] thus remains open. Similarly, it is not yet clear if the behavior of the random continuously symmetric system in $d=4+\epsilon$ dimensions is the same as that of its pure counterpart in $d=2+\epsilon$ dimensions.[9]

Another heuristic approach to critical phenomena of Ising systems was recently proposed by Thompson.[10] His basic assumption is, that all the three first terms in Eq. (1), for a volume L^d, are separately of order 1. For such a volume, the new fourth term in (1) is of order[1] $\lambda^{1/2}L^{d/2}M$. Since, from perturbation theory, the variable $u\lambda$ replaces u everywhere,[1] it is reasonable to replace Thompson's assumption concerning the third term in (1) by the assumption that only its product with the square of the last term is of order unity. Following Thompson's other steps this immediately yields $\nu = \frac{1}{2}$ for $d > 6$, and $\nu = (d+2)/4(d-2)$ for $d < 6$. Note that his yields $\nu \to \infty$ as $\nu \to d_a = 2$, in agreement with the heuristic predictions of Ref. 1 (this assumption was not fed into the theory here!). The ϵ expansion of this result near $d=6$ is not identical with that of the pure case near $d=4$, but this is probably not to be expected (it breaks down at order ϵ^2 for the pure case[10]). It is interesting to note that this approximation gives $\nu = \frac{5}{4}$ at $d=3$, instead of $\nu = \frac{5}{8}$ for the pure system![10] Since $d=3$ is much closer to d_a in the random case, one would expect a much larger deviation from mean field, compared to the pure case, even if the Thompson-like estimate is not very accurate. Thus, experimental (e.g., on displacive or charge-density-wave transitions, with randomly frozen impurities or charges), numerical (e.g., with use of Monte Carlo techniques) and further theoretical investigations of this case are extremely interesting. We note that the model discussed here is also equivalent to some spin-glasses, in uniform magnetic fields.[11]

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ERRATA

PRIMORDIAL SUPERHEAVY ELEMENT 126. C. Y. Wong [Phys. Rev. Lett. 37, 664 (1976)].

Page 665, second column, twelfth line: "=2.26" should read "=2.66 MeV." Page 666, first column, twelfth line: "both the neutron and the proton" should read "the proton."

ETCH INDUCTION TIME IN CELLULOSE NITRATE TRACK DETECTORS. F. H. Ruddy, H. B. Knowles, and G. E. Tripard [Phys. Rev. Lett. 37, 826 (1976)].

The first sentence in the second paragraph in the second column on page 828 should read: It is remarkable that, to a first approximation, the surface effect produced by these heavy ions as they enter CN films appears to depend on $[(dE/dx)_{W<W_0}]/Z^*$ which thus depends only on the first power of the effective charge, Z^*.