Spectral Distribution of Drift-Wave Fluctuations in Tokamaks

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(Received 26 February 1976)

An expression for the spectral distribution of drift-wave fluctuations is obtained from a new mode-coupling equation that describes the superposition of drift-wave normal modes induced by the divergence of the nonlinear $\vec{E}\times\vec{B}$ convective flux. Computations for the adiabatic toroidal compressor experiment indicate that the observed spectrum is consistent with the theoretical spectrum.

Previous drift-wave turbulence calculations have served mainly to develop an estimate for the net fractional turbulent energy density summed over modes at saturation, as obtained from several different considerations. These include the energy density released by profile flattening; the limit of linear motion when convection through one wavelength occurs during one wave period; and the limit of a turbulent damping, estimated by $k_\perp^2 D_\perp$, comparable to the linear growth rate. The several approaches generally give a consistent description of the magnitude and scaling of the total turbulent energy density and, equivalently, the average squared density fluctuation for drift-wave type instabilities. Recent measurements by Mazzucato[1] with microwave scattering from small-scale density fluctuations in the adiabatic toroidal compressor (ATC) have provided a direct, nonperturbing measurement of the wave-number spectrum for drift-wave fluctuations at six values of the azimuthal wave number, ranging from $k_y\rho \cong 0.3$ to 1.5, where $\rho = c(m_i T_e)^{1/2}eB$. In the course of a theoretical analysis of this experiment I have developed a mode-coupling theory for the shear-controlled drift-wave normal modes. From the theory I obtain for the stationary spectral distribution of drift-wave fluctuations an expression that appears to agree well with the experiment. The expression is consistent with the previous theoretical results for the total turbulent energy density summed over the wave spectrum.

In this Letter I consider the nonlinear stabilization of the drift modes introduced by the divergence of the convective $\vec{E}\times\vec{B}$ flux in the ion-fluid continuity equation. Two physically distinct nonlinear effects are introduced by this flux. One effect arises from the Burgers or Korteweg-de Vries type of nonlinearity[2] that produces a steepening of the wave form in its dependence on the phase variable $\zeta = k_y y - \omega_k t$. The magnitude of this wave-form distortion is readily calculated at sufficiently small amplitudes, and I estimate that the steepening is a subdominant effect in the experiment under discussion. The second effect is the diffusive or viscous damping in a wave of given $\vec{k}, \omega_{\vec{k}}$ induced by the presence of the other waves. The latter effect appears in the theory as the result of the nonlinear Doppler shifts in the given wave due to the motion of the fluid from the other waves. The nonlinear diffusive effect derived here is more complicated than has been indicated by previous theories[3] that shift $\omega_{\vec{k}}$ to $\omega_{\vec{k}} + ik_\perp^2 D_\perp$, with $D_\perp$ determined by the well-known consistency equation. In the present theory I retain certain secular terms arising from the small-amplitude perturbation expansion of the nonlinear ion-fluid equation, where the identification and summation of higher order terms may be understood in terms of the methods used earlier for Vlasov turbulence by Choi and Horton.[4]

Previous kinetic theory treatments of linear drift waves indicate that the ion response is reasonably well approximated as hydrodynamic for $T_e \gtrsim 3T_i$ and $|\eta_i| = |d\ln T_i/d\ln n| \lesssim 1$, as is the case for the experiment. As explained previously, we neglect as a subdominant effect the steepening nonlinearities due to convective derivatives and the electron temperature gradient. We obtain the equation for the ion density $n_i(\vec{x}t)$ in terms of field $\vec{E} = -\nabla\varphi(\vec{x}t)$:

$$\frac{\partial n_i(\vec{x}t)}{\partial t} + \nabla\cdot\left(\frac{cn_i\vec{B}\times\nabla\varphi}{B^2}\right) - \nabla_\perp\cdot\left(\frac{c^2 n_i m_i}{eB^2}\nabla_\perp\frac{\partial\varphi}{\partial t}\right) - \vec{B}\cdot\nabla\left(\frac{en_i\vec{B}}{m_i B^2}\cdot\nabla\int^t \varphi(\vec{x}t')dt'\right) = 0. \tag{1}$$

We introduce $\tilde{n}, \tilde{\varphi}$ for the scaled, finite-amplitude variables by writing

$$n_i(\vec{x}t) = n_0(x)[1 + \tilde{n}(\vec{x}t)], \quad eQ(\vec{x}t) = T_e(x)\tilde{\varphi}(\vec{x}t),$$

where $\tilde{n}$ is the local fractional density fluctuation and $\tilde{\varphi}$ is the potential fluctuation measured with respect to the local temperature $T_e(x)$. We seek solutions of Eq. (1) for $\tilde{n}_i = \tilde{n}_i(\tilde{\varphi})$ for $\tilde{\varphi}_{\rm rms} = \langle \tilde{\varphi}^2 \rangle^{1/2} \sim \epsilon \ll 1$. The electron response is taken as linear and is obtained from the kinetic-theory calculation of Horton[5] that uses the Lorentz collision operator and a three-energy-range approximation for the nonadiabatic electron response. The linear approximation for the electrons is justified *a posteriori* by estimating the electron scattering at the turbulence levels obtained from the fluid mode coupling. In the intermediate electron energy range, for example, the linearization follows from the fact that electron trapping velocity is found to be small compared to the electron velocity.

Separating out the linear operator $\hat{L}_i \tilde{\varphi}$ in Eq. (1) and introducing the scaled field variables, we obtain

$$i\,\frac{\partial \tilde{n}}{\partial t} = \hat{L}_i \tilde{\varphi} + D_B\left(\frac{\partial \tilde{n}}{\partial x}\frac{\partial \tilde{\varphi}}{\partial y} - \frac{\partial \tilde{n}}{\partial y}\frac{\partial \tilde{\varphi}}{\partial x}\right), \tag{2}$$

with the linear ion fluid operator $\hat{L}_i$ defined by

$$\hat{L}_i \tilde{\varphi} = [\omega_{*e} + k_\parallel^2 c_s^2/\omega + \omega\rho^2(-k_y^2 + \partial_x^2)]\,\tilde{\varphi}_k(x)\exp(ik_y y + ik_z z - i\omega t) + \text{c.c.}, \tag{3}$$

where $\omega_{*e} = -(k_y c T_e/eB)(d\ln n_0/dr)$, $\rho = c(m_i T_e)^{1/2}/eB$, and $D_B = cT_e/eB$. The eigenfunctions[6] of the ion response $\hat{L}_i u_k(x) = L_k(\omega) u_k(x)$ are localized about the rational surfaces x_k where $\vec{k}\cdot\vec{B} = k_\parallel(r = x_k)B = 0$. To a good approximation[6] the eigenfunctions are given by $u_k(x) = [(i\sigma_k)^{1/2}/2^n n!\pi^{1/2}]^{1/2} H_n[(i\sigma_k)^{1/2}\xi] \times \exp(-i\sigma_k \xi^2/2)$, where $\xi = x - x_k$ and $\sigma_k = |k_y| c_s/\omega\rho L_s$ with $\sigma_{-k} = -\sigma_k^*$. We proceed to expand the density and potential fluctuations in a time-dependent superposition of normal modes,

$$\tilde{\varphi}(\vec{x}t) = \textstyle\sum_k \{\hat{\varphi}_k(t) u_k(x)\exp(ik_y y + ik_z z - i\omega_k t) + \hat{\varphi}_{-k}(t) u_{-k}(x)\exp(-ik_y y - ik_z z + i\omega_k^* t)\},$$

with $\hat{\varphi}_{-k}(t) = \hat{\varphi}_k^*(t)$, and to neglect the harmonic components generated by the self-interaction in the normal modes. Assuming that the frequency spectrum

$$\tilde{\varphi}_k(\omega) = \int_{-\infty}^{+\infty} dt\, \exp[i(\omega - \omega_k)t]\,\hat{\varphi}_k(t)$$

of the normal-mode amplitudes remains peaked about $\omega \simeq \omega_k$, we obtain the following equation for the time dependence of $\hat{\varphi}_k(t)$:

$$\left[i\omega_k \frac{\partial D_k^R}{\partial \omega_k}\frac{\partial}{\partial t} + i\omega_k D_k^I(\omega_k) - D_B^2 \sum_q \frac{M(k,q)|\hat{\varphi}_q(t)|^2}{\omega_k - \omega_q + i\nu_k}\right]\hat{\varphi}_k(t)$$
$$= D_B \sum_q N(k,q)\left(\frac{L_q(\omega_q)}{\omega_q} - \frac{L_{k-q}(\omega_{k-q})}{\omega_{k-q}}\right)\hat{\varphi}_q(t)\hat{\varphi}_{k-q}(t)\exp[-i(\omega_q + \omega_{k-q} - \omega_k)t], \tag{4}$$

where

$$M(k,q) = \tfrac{1}{2}\int_{-\infty}^{+\infty}(q\dot{u}_k u_{-q} + k u_k \dot{u}_{-q})(q\dot{u}_k u_q + k u_k \dot{u}_q)\,dx, \quad N(k,q) = -\tfrac{1}{2}\int_{-\infty}^{+\infty} u_k[q\dot{u}_{k-q}u_q - (k-q)u_{k-q}\dot{u}_q]\,dx, \tag{5}$$

with $\dot{u}_k = du_k/dx$ and $D_B = cT_e/eB$. In Eq. (4) $D_k^R(\omega)$ and $D_k^I(\omega)$ are the real and the imaginary parts of the dispersion relation where, with the fluid approximation for the ions and the kinetic theory calculation of Ref. 5 for the nonadiabatic part $H_{ek}(\omega_k)$ of the electron response, we have

$$D_k(\omega_k) = 1 + k_y^2\rho^2 - \omega_*/\omega_k - H_{ek}(\omega_k) + iS(1+2n)|\omega_{*e}|/\omega_k = 1 - H_{ek}(\omega_k) - L_k(\omega_k)/\omega_k,$$

where $L_k(\omega_k)$ is the eigenvalue of the ion response. The linear growth rate of the lowest radial mode, which is given by $\gamma_k = -\operatorname{Im} D_k^I(\omega_k)/[\partial D_k^R(\omega_k)/\partial\omega_k]$ with $n = 0$, has a maximum between $k_y\rho = 0.5$ and 1.0 depending on the magnitude of η_e and a cutoff due to ion Landau damping for $k_y\rho$ below $k_y\rho \simeq 2(2T_i/T_e)^{1/2}(r_n/L_c)$. Ion Landau damping is the microscopic damping mechanism that allows a steady-state energy balance to be achieved in the saturated turbulent state. The damping is present both at low $k_y\rho$ and at $k_y\rho \sim 1$ for $k_\parallel(x)c_s \sim \omega_k$, which are regions accessible through mode coupling and radial wave propagation. The mode-coupling terms on the right-hand side of Eq. (4) are of the same form as those derived by Sagdeev and Galeev[7] from kinetic theory. Since the dispersion essentially forbids the resonant interaction $\omega_q + \omega_{k-q} = \omega_k$ and since in the linear approximation $L_k(\omega_k) \approx \omega_k$, it appears justified to neglect the weak mode-coupling interactions induced by the terms proportional to $N(k,q)$. In this case nonlinear saturation is obtained by balancing the terms of the left-hand side of Eq. (4).

To reduce the mode-coupling equation further we note that the relative spacing of the azimuthal modes ($k_y=m/r$ and hence $\Delta k_y/k_y=1/k_y r\sim\rho/r$) is sufficiently dense to permit the introduction of the continuous spectral density $I(k_y\rho)$ and replace the sum over modes with integrals. We define the normalized spectral density of the potential fluctuations by

$$\left\langle\frac{e^2\varphi^2(\vec{x}t)}{T_e^{\,2}}\right\rangle=\langle\tilde{\varphi}^2\rangle=\int_0^\infty \rho\,dk_y I(k_y\rho), \tag{6}$$

where $I(k_y\rho)$ is then related to $|\hat{\varphi}_k(t)|^2$ by,

$$\rho I(k_y\rho,t)=(2e^2/T_e^{\,2})(2\pi)^{-3}\iint V\,dk_x dk_z|\hat{\varphi}_k(t)|^2,$$

where $V=2\pi^2r^2R$ is the volume of the torus. In addition we reduce the equation further by calculating the matrix element $M(k,q)$ with sinusoidal approximations for $u_k(x)$ and $u_q(x)$ with the radial wave numbers k_x and q_x taken from the average radial wave numbers of the eigenfunctions. With these approximations the stationary spectrum becomes the solution of

$$\omega_k D_k^{\,I}(\omega_k)-\tfrac{1}{2}\bar{k}_x^{\,2}D_B^{\,2}\,\mathrm{Im}\int_0^\infty\frac{\rho\,dq_y(k_y^{\,2}+q_y^{\,2})I(q_y\rho)}{\omega_{k_y}-\omega_{q_y}+i\nu_k}=0, \tag{7}$$

for k regions where $D_k^{\,I}(\omega_k)<0$, and $I(k_y\rho)\cong I_{\rm th}(k_y\rho)$ for stable regions where $D_k^{\,I}(\omega_k)>0$. The thermal fluctuation level is readily estimated using $\bar{k}_\parallel r_n=k_y\rho S^{1/2}$ to obtain $I_{\rm th}(k_y\rho)\simeq k_y\rho S(n_e\rho^2r_n)^{-1}\times(1+k_y^{\,2}\rho^2)^{-1}$. In the present case $I_{\rm th}\approx10^{-16}$ which implies a time of order $13/\gamma_{\max}$ or a few tenths of a millisecond to reach saturation at $I\approx10^{-5}$.

In Eq. (7) the quantity ν_k describes the finite correlation time of the $\omega_k=\omega_q$ resonance. The decorrelation rate ν_k is proportional to the turbulence level which is low for the systems of interest. We assume that the spectrum $I(k_y\rho)$ is broad compared to the resonance width, which allows $I(k_y\rho)$ to be obtained independently of the exact form of ν_k. For an estimate of ν_k we obtain from the simply renormalized theory of Choi and Horton[4] that $\nu_k\simeq\omega_k\bar{k}_x^{\,2}r_n^{\,2}k_y\rho I(k_y\rho)$. For $\nu_k\ll\omega_k$ and a broad spectrum, Eq. (7) reduces to

$$\frac{\pi\bar{k}_x^{\,2}k_y^{\,2}D_B^{\,2}\rho I(k_y\rho)}{|d\omega_{ky}/dk_y|}=-\omega_k D_k^{\,I}(\omega_k), \tag{8}$$

where $D_k^{\,I}(\omega_k)<0$. With the further approximations that $\bar{k}_x\rho=S^{1/2}$ and $|d\omega_{ky}/dk_y|=v_{de}$, the result may be written as

$$I(k_y\rho)=\frac{W_f}{n_eT_e}\,\frac{-D_k^{\,I}(\omega_k)}{\pi k_y\rho}, \tag{9}$$

where $W_f=n_eT_e(\rho/r_n)^2(L_s/r_n)$ is the maximum fractional turbulent energy density available to the instability at saturation. The spectrum peaks on the long wavelength side of the maximum growth rate and returns to essentially the thermal fluctuation level in stable regions. The electron fluctuations are approximately equal to the potential fluctuations, and the turbulent energy density W is given approximately by

$$\frac{W}{n_eT_e}=\left\langle\frac{\tilde{n}^2}{n_0^{\,2}}\right\rangle=\langle\tilde{\varphi}^2\rangle=\frac{W_f}{n_eT_e}\int_{\Delta k}\frac{dk_y}{\pi k_y}[-D_k^{\,I}(\omega_k)], \tag{10}$$

where Δk is the width of the unstable k_y region. This result for W/n_eT_e agrees with earlier theoretical calculations which indicate that in fully developed strong turbulence W_f is the maximum turbulence level. For weaker turbulence the result indicates that $W\simeq W_f(\Delta k_y/\pi k_y)|\max(-D_k^{\,I})|$ can be considerably less than W_f.

In recent microwave scattering experiments on ATC, Mazzucato[1] measured the spectral distribution of electron density fluctuations for frequencies and wavelengths in the range of drift waves. From the observed scattered microwave power Mazzucato uses scattering theory to infer the value of $\langle|n_e(k)|^2\rangle$ for six values of k where k is essentially the azimuthal mode number k_y used in the above analysis. The total mean fluctuating density in his work is given by $\langle\tilde{n}^2\rangle=\int_0^\infty\langle|n(k)|^2\rangle\,dk$. In terms of the scaled spectral density $I(k_y\rho)$ that is defined in the above analysis we have that $\rho I(k_y\rho)\simeq\langle|n(k_y)|^2\rangle/n_e^{\,2}$. In Fig. 1 I have repeated the result reported by Mazzucato and have added the result of my calculation from Eq. (9) for that experiment. For the calculation of $\mathrm{Im}\boldsymbol{H}_{ek}(\omega_k)$ I use the three-energy-range approximation explained in Ref. 5. In evaluating the theoretical quantities I have digitized the reported density and electron temperature profiles and calculated the drift wave parameters at $r=6.4$, 8.0, 9.6,

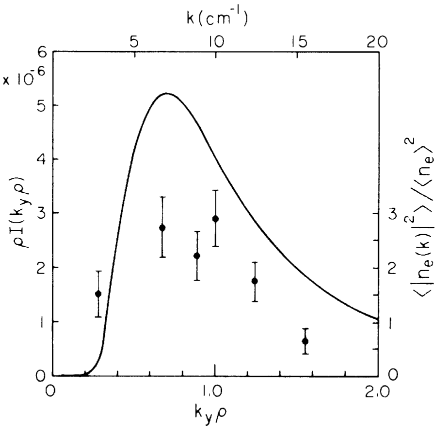

FIG. 1. Comparison of the theoretical spectral distribution $I(k_y\rho)$ given by the solid curve, with the previously reported data (Ref. 1) for the electron density fluctuation spectrum obtained from microwave scattering in the ATC experiment.

and 11.2 cm assuming that the current density $j_{\parallel}(r)$ varies at $T_e^{3/2}(r)$. Similar evaluations for other tokamak experiments are being performed and will be reported in a later article.

In conclusion, the theory appears to explain several features of both the spectral distribution and the total mean density fluctuation observed in the experiment. We observe that the theory is a first-principles calculation in that, although numerous conventional approximations are made, there are no free parameters available.

The author wishes to thank Dr. E. Mazzucato for making the microwave scattering data available prior to publication and for his useful comments concerning the experiment. He also thanks Dr. R. Estes and Mr. Kwak for their work on the calculations.

*Work supported by the U. S. Energy Research and Development Administration Contract No. E-(40-1)-4478.

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Alfvén-Wave Heating in the Proto-Cleo Stellarator*

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Global excitation of Alfvén waves in the Proto-Cleo $l = 3$ stellarator was accomplished by exciting a helical winding corresponding to a $q = 3$ rational surface with a pulsed, high-power rf source. A doubling of both the electron and ion temperatures was observed, and a slight increase in the ratio of the temperatures with and without rf heating occurred at the predicted resonance locations. Enhanced loss also occurred during heating, with 2.5-kHz oscillations observable in a microwave interferometer signal after heating.

Alfvén-wave heating of toroidally confined plasmas has been proposed by several authors.[1-3] Recent experimental work at Kyoto University[4] and at Kharkov[5] apparently show that *local* excitation of Alfvén waves leads to heating of both ions and electrons, when the resonant condition $\omega = k_{\parallel} V_A$ was satisfied. V_A is the Alfvén speed and $k_{\parallel}$ is the wave number in the direction parallel to the dc magnetic field. $k_{\parallel}$ was always greater than $2\pi L$ where L is the distance around the torus. It is the purpose of this Letter to report successful Alfvén-wave heating of plasma contained