PHYSICAL REVIEW LETTERS

VOLUME 37 | 8 NOVEMBER 1976 | NUMBER 19

Quantum Gravity and World Topology*

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(Received 20 September 1976)

The path integral of quantum gravity probes space-times with nontrivial topology and induces chiral symmetry breaking via an anomaly proportional to $\epsilon^{\mu\nu\rho\sigma}R^{\alpha}{}_{\beta\mu\nu}R^{\beta}{}_{\alpha\rho\sigma}$ in the divergence of the axial fermion number current. The corresponding classical instanton solution of Einstein's equations with cosmological term is found. The Euler-Poincaré characteristic and ordinary instanton number are also discussed.

Non-Abelian gauge theories exhibit interesting topological invariants.[1] One of these, instanton number,[2] appears to play the central role[3] in symmetry breaking via Adler-Bell-Jackiw (ABJ) anomalies.[4] These are related to a (quantum) tunneling between topologically inequivalent vacua. Here we explore the topological invariants of gravitation theory—the gauge theory of the Lorentz group—and their connection with anomalies. We find an additional gravitational term in the ABJ axial baryon-number anomaly, as well as a further topological invariant apparently unrelated to anomalies. We find complex projective two-space $P_2(C)$ (two complex dimensions, i.e., four spacelike real dimensions) to be a solution of the classical Einstein equations with cosmological term, that plays the role of a gravitational instanton for the ABJ anomaly. Finally we reconsider the original Belavin-Polyakov-Schwartz-Tyupkin (BPST) instanton solution[2] in a general relativistic setting.

In the O(4) gauge theory there are two kinds[2] of instanton number $\{\pi_3(\mathrm{O}(4))=Z\times Z\}$:

$$\tau_{\mathrm{O}(4)}=\int FF^*\,d^4x,\quad \chi_{\mathrm{O}(4)}=\int FF^{**}\,d^4x,$$

$$FF^*=-(1/96\pi^2)\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}{}^{\alpha\beta}F_{\rho\sigma}{}^{\beta\alpha},$$

$$FF^{**}=(1/128\pi^2)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma\delta}F_{\mu\nu}{}^{\alpha\beta}F_{\rho\sigma}{}^{\gamma\delta},$$

where α, β, γ, and δ are internal O(4), and μ, ν, ρ, and σ are Lorentz indices, respectively. Einstein theory viewed as a Lorentz gauge theory suggests the analogy[5]

$$\mathrm{O}(4)\text{ gauge theory}\leftrightarrow\text{gravitation},$$

$$F_{\mu\nu}{}^{\alpha\beta}\leftrightarrow R^{\alpha}{}_{\beta\mu\nu},$$

where $R^{\alpha}{}_{\beta\mu\nu}$ is the Riemann-Christoffel tensor. The analogs of $\tau_{\mathrm{O}(4)}$ and $\chi_{\mathrm{O}(4)}$ are the quantities,

$$\tau=\int RR^*\,d^4x,\quad \chi=\int RR^{**}\,d^4x,$$

$$RR^*=\frac{-1}{96\pi^2}\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{g}}R^{\alpha}{}_{\beta\mu\nu}R^{\beta}{}_{\alpha\rho\sigma}\sqrt{g},\tag{1}$$

$$RR^{**}=\frac{1}{128\pi^2}\frac{\epsilon^{\alpha\beta\gamma\delta}}{\sqrt{g}}\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{g}}R_{\alpha\beta\mu\nu}R_{\gamma\delta\rho\sigma}\sqrt{g}.$$

χ is recognized as the Euler-Poincaré characteristic (related to the handle number), and τ as the index of the four-dimensional space-time manifold.[6]

The integrands RR^*, RR^{**}, FF^*, and FF^{**} share the following important features: (i) They are exact divergences[2,7] (for the handle number density this is appropriately known as the Bach identity[7]). (ii) They have dimension of (length)$^{-4}$ so that their integrals are dimensionless numbers (integers for compact manifolds). (iii) They are singlets of all internal symmetries (color, flavor). (iv) RR^*, FF^*, and FF^{**} are pseudoscalar [$\epsilon^{\alpha\beta\gamma\delta}$ in the expression of FF^{**} is invariant under space inversions as it has internal O(4) indices], but RR^{**} is scalar, and all four quantities are even under charge conjugation. In addition, RR^* can be expressed in terms of the Weyl conformal tensor so that τ is Weyl invariant.[8]

We now ask the main question: To what anomalies do RR^* and RR^{**} contribute? In other words, are there currents conserved in the classical field theory, such that in the quantum theory their divergence equals RR^* or RR^{**}?

We start with RR^*. Its quantum numbers are such [(iii) and (iv) above] that it can only contribute to an anomaly in the divergence of the axial fermion-number current $j^{5\mu}$ as in the ABJ case. We thus try

$$\partial_\mu j^{5\mu} = \partial_\mu(\sqrt{g}\, h^{\mu a}\,\bar{\psi}\gamma_a\gamma_5\psi) = rRR^*. \tag{2}$$

Here $h^{\mu a}$ is the gravitational *Vierbein.* The space-time integral of $\partial_\mu j^{5\mu}$ is ΔQ^5, the change of the axial fermion charge: an integer. The space-time integral of RR^* is the index of the space-time manifold which for compact manifolds is also an integer. Hence the proportionality constant r must be a *rational* number.

To see whether Eq. (2) works in a way similar to the ordinary ABJ anomaly we have coupled the gravitational field to a fermion field and have computed the anomaly in the axial-vector Ward identity in the case of two external gravitons. In addition to the fermion triangle diagram, there is here also a second diagram in which the two gravitons are emitted from the same point of the fermion loop by a seagull vertex. We have made use of the regulator method which automatically maintains gauge invariance (general covariance) but possibly jeopardizes the axial-vector current conservation. After a straightforward computation we in fact find an anomaly,

$$\partial_\mu j_\mu{}^5 = \tfrac{1}{4}RR^*, \tag{3}$$

or $r=\frac{1}{4}$ in Eq. (2) (for N fermions in the theory, $r=\frac{1}{4}N$). An instanton interpretation of this value of r will be given below. The above expression contains the contributions from diagrams made of all possible graviton trees attached to a single fermion loop.

Discussions on possible radiative corrections to the anomaly in Eq. (3) are necessarily obscured by the absence of a renormalizable theory of quantized gravity. We can provide, however, a formal and nonrigorous argument which suggests that to all orders in perturbation theory, the anomaly should not be renormalized. We first note that since RR^* is a total divergence, we have

$$\int d^4z\; \delta[RR^*(z)]/\delta g_{\alpha\beta}(x) = 0. \tag{4}$$

Hence the correction to the anomaly from one additional internal graviton line, for instance, vanishes:

$$\int d^4z \int d^4x \int d^4y\, D(x-y)_{\alpha\beta,\alpha'\beta'} \frac{\delta^2[RR^*(z)]}{\delta g_{\alpha\beta}(x)\delta g_{\alpha'\beta'}(y)} = 0, \tag{5}$$

where $D(x-y)$ is the graviton propagator. On the other hand, using generalized unitarity (or the tree theorem) multiloop processes can be computed in a gauge invariant manner by an appropriate "sewing" procedure from lower-order loop diagrams. The procedure always involves functional differentiations and hence by mathematical induction all higher loop corrections will vanish.

To see the topological origin of the anomaly (3) let us find a classical solution of Einstein's equations with cosmological term that has index $\tau=1$. Complex projective two-space $P_2(C)$ has $\tau=1$ and admits a Kaehler metric, as noted by Fubini at the turn of the century.[10] The corresponding four-dimensional real Riemannian [signature $(++++)$] manifold has the metric,

$$g_{\mu\nu} = \frac{4a^2}{a^2+x^2}\left(\delta_{\mu\nu} - \frac{x_\mu x_\nu + \tilde{x}_\mu\tilde{x}_\nu}{a^2+x^2}\right), \tag{6a}$$

where a is a constant length, and

$$x_\mu = \delta_{\mu\nu}x^\nu, \quad \tilde{x}_\mu = C_{\mu\nu}x^\nu,$$
$$x^2 = \delta_{\mu\nu}x^\mu x^\nu, \quad C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{6b}$$

Starting from Eqs. (6) one readily calculates the curvature tensor and inserting into Eqs. (1) finds that in this case $\tau=1$, $\chi=3$ (by contrast spherical de Sitter space has $\tau=0$, $\chi=2$). Remarkably, we find the metric (6) to be a solution of Einstein's equation (in vacuum) with cosmological constant $\Lambda = 3/2a^2$. For every value of the parameter a there is a solution. This is parallel to the BPST solution of ordinary Yang-Mills theory, as is the appearance of the factor $a^2/(a^2+x^2)$ in Eq. (6). The solution (6) of Einstein's equations plays the role of a gravitational instanton. The corresponding anti-instanton is obtained by leaving the metric (6) unaltered, but changing the orientation (the *Vierbein*) of the space-time manifold. The large-x behavior of this instanton solution is $g_{\mu\nu} \to 0$, $R^{\alpha}{}_{\beta\mu\nu} \to 0$, $R_{\mu\nu} \to 0$, but R = const since $g^{\mu\nu} \to \infty$ [the manifold $P_2(C)$ is actually compact!]. The mechanism leading to the anomaly (3) appears to be the same as in the ABJ case. We have not carried out the corresponding tunneling calculations, but we note that the tunneling amplitude contains the factor $\exp(-\text{const}\times a^2/G)$ so that small values of a dominate.

One may be intrigued by the fact that unlike the ordinary ABJ anomaly, where the coefficient of FF^* is the integer 2, the coefficient of RR^* in Eq. (3) is fractional. This means that the actual tunneling due to the anomaly (3) occurs in the $4n$-instanton sectors ($n = 1, 2, 3, \ldots$) of the theory.

We now consider the other topological invariant of the gravitational field, the Euler-Poincaré characteristic χ. Its density RR^{**} [Eq. (1)] does not seem to lead to any anomalies. Indeed in view of its quantum numbers ($J^{PC} = 0^{++}$) the only candidate is an anomaly in the divergence of the Weyl scale current. But the relevant piece of this current is a c-number quantity and hence leads to no quantum anomalies. The essential c-number nature of the Weyl current is most readily seen by writing down a Weyl-invariant gravitational Lagrangian [e.g., Eq. (7) below] and coupling it to fermions. The contribution of the gravitational and φ fields to the Weyl current can be gauged away altogether, while the contribution of Fermi fields (even under charge conjugation!) is an anticommutator $\gamma_{\mu\alpha\beta}\{\bar{\psi}_\alpha, \psi_\beta\}$ (rather than the usual commutator) and hence a c-number. Additional scalar fields may give q-number contributions to the Weyl current but this is inconsequential as they produce no anomaly.

We now briefly note the effects of general relativity on the BPST-instanton solution of classical Yang-Mills theory in flat space. To this effect consider the action,

$$A = \int d^4x \sqrt{g}\,[-(\varphi^2/12)R + \tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \lambda\varphi^4 - \tfrac{1}{4}F_{\mu\nu}{}^i F^{\mu\nu i}], \qquad (7)$$

φ being a scalar field, and $F_{\mu\nu}{}^i$ the SU(2) Yang-Mills fields. The action A is invariant under Weyl transformations[8]

$$g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu}, \quad \varphi \to \Omega^{-1}(x)\varphi,$$
$$F_{\mu\nu}{}^i \to F_{\mu\nu}{}^i, \quad x_\mu \to x_\mu,$$

that depend on the arbitrary function $\Omega(x)$. By a Weyl transformation we can force φ to take the constant value $(3/4\pi G)^{1/2}$, where G is Newton's gravitational constant. In this gauge the action A is identical to that of Einstein-Yang-Mills theory with cosmological constant $\Lambda = 9\lambda/2\pi G$. The field equations derived by varying the action A admit the obvious solution,

$$g_{\mu\nu} = \delta_{\mu\nu}, \quad A_\mu{}^i = (A_\mu{}^i)_{\text{BPST}},$$
$$\varphi = \varphi_{\text{F}} = (2b^2/\lambda)^{1/2}(x^2 + b^2)^{-1},$$

where $(A_\mu{}^i)_{\text{BPST}}$ denotes the ordinary BPST solution and φ_{F} is Fubini's solution[11] of flat-space $\lambda\varphi^4$ theory. From this solution, the Weyl transformations generate new solutions for every conformally flat space-time, in particular for de Sitter space.

To sum up, our main result is that the path integral of quantum gravitation probes space-times with nontrivial topology ($\tau \neq 0$) and thereby induces chiral symmetry breaking via an anomaly [Eq. (3)] in the divergence of the axial fermion-number current. This provides an affirmative answer to Chern's old question[6] as to whether τ "could . . . have some use in physics."

We derived much stimulation from a discussion with Dr. G. 't Hooft in the early phases of this work. We have profited from interesting discussions about characteristic classes with Professor R. Lashof, and enjoyed numerous valuable conversations with Mr. John Sidles especially on the general relativistic BPST instanton. After the completion of this work, reports by G. Domokos [DESY Report No. DESY 76/24 (unpublished)] and F. Wilczek [Princeton Report No. 1976 (unpublished)] have reached us. Domokos treats Fubini's solution φ_{F} in a Weyl-invariant general relativistic setting. His discussion is a special case of that at the end of this paper. Wilczek explores the topology of O(4) gauge theory and of gravitation theory with a very different emphasis and little overlap in results with us.

Note added.—Since submitting this paper, we have become aware of work by R. Delbourgo and A. Salam, Phys. Lett. 40B, 381 (1972), in which the gravitational correction to partial conservation of axial-vector currents is calculated but a result differing by a factor $\frac{1}{2}$ from our Eq. (3) is found. Also a paper by A. A. Belavin and D. E. Burlankov, Phys. Lett. 58A, 7 (1976) deals with RR^* and RR^{**} but with an emphasis quite different from ours. We thank Dr. Judy Lieberman, Professor Heinz Pagels, and Professor Roman

Jackiw for calling these references to our attention.

*Work supported in part by the National Science Foundation, Contract No. PHYS74-08833, and the Louis Block Fund, the University of Chicago.

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Radio-Frequency Atomic Beam Measurement of the $(2^2S_{1/2}, F=0)$-$(2^2P_{1/2}, F=1)$ Lamb-Shift Interval in Hydrogen

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(Received 23 June 1976)

The $(2^2S_{1/2}, F=0)$-$(2^2P_{1/2}, F=1)$ interval in atomic hydrogen has been measured in zero magnetic field by scanning a radio-frequency perturbation through the atomic resonance. The measured interval was found to be 909.904 ± 0.020 MHz, which is consistent with that reported by Lundeen and Pipkin, 909.940 ± 0.020 MHz. Our deduced value of the Lamb shift 1057.862 ± 0.020 MHz is in good agreement with Mohr's calculated value 1057.864 ± 0.014 MHz but not with Erickson's value of 1057.912 ± 0.011 MHz.

In this Letter we report a measurement of the $(2^2S_{1/2}, F=0)$-$(2^2P_{1/2}, F=1)$ interval in atomic hydrogen and from it deduce a value of the hydrogen $n=2$ Lamb shift. Our method differs from that of a previous determination of Lundeen and Pipkin[1] in that we have used a slower beam (21 keV) and a single microwave region in the form of a 50-Ω transverse transmission line to induce the transition.

The separated-field[2] approach of Lundeen and Pipkin would appear to have an important advantage over the single-field technique[3] because it produces an interference signal whose linewidth can be made significantly narrower than that obtained by the single-field method. However when one of the atomic states can decay by spontaneous radiation, "interference narrowing" of the resonance is obtained at the expense of signal strength as can be seen in Fig. 3 of Ref. 1 in which Lundeen and Pipkin report a threefold narrowing. The significant loss in signal strength and increased complexity of the line shape and apparatus are major disadvantages which should be weighed against the advantages of having a much reduced linewidth. In our experiment we have chosen to use a single-field region in the form of a 50-Ω transverse slab line[4] and a slower beam (21 keV) which together ensure an adiabatic switch-on and -off of the perturbation. Since the solutions of Maxwell's equations for slab-line geometry (see Ref. 4) can be obtained analytically, a precise description of the spatial distribution of the field exists, so it is possible to give an accurate description for the atomic and instrumental line shape.

Figure 1 shows the main features of the apparatus in which the metastable hydrogen beam is produced by charge exchange on molecular hydrogen of a 100-μA 21-keV proton beam extracted from a radio-frequency ion source.[5] Noting that the natural linewidth of the $2P$ state is ≈ 100 MHz then, from Fig. 2, it can be seen that an oscillator tuned to about 1120 MHz will simultaneously drive both the β and γ resonances, thus quenching the $2^2S_{1/2}, F=1$ level by $\Delta M_F=0$ transitions. Radio-frequency state selection of the metastable